Inertial forces and the foundations of optical geometry

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Abstract. Assuming a general timelike congruence of worldlines as a reference frame, we derive a covariant general formalism of inertial forces in General Relativity. Inspired by the works of Abramowicz et. al. (see e.g. Abramowicz and Lasota 1997 *Class. Quantum Grav.* **14** A23-30), we also study conformal rescalings of spacetime and investigate how these affect the inertial force formalism. While many ways of describing spatial curvature of a trajectory has been discussed in papers prior to this, one particular prescription (which differs from the standard projected curvature when the reference is shearing) appears novel. For the particular case of a hypersurface-forming congruence, using a suitable rescaling of spacetime, we show that a geodesic photon is always following a line that is spatially straight with respect to the new curvature measure. This fact is intimately connected to Fermat's principle, and allows for a certain generalization of the optical geometry as will be further pursued in a companion paper (Jonsson and Westman 2006 *Class. Quantum Grav.* **23** 61). For the particular case when the shear-tensor vanishes, we present the inertial force equation in three-dimensional form (using the bold face vector notation), and note how similar it is to its Newtonian counterpart. From the spatial curvature measures that we introduce, we derive corresponding covariant differentiations of a vector defined along a spacetime trajectory. This allows us to connect the formalism of this paper to that of Jantzen et. al. (see e.g. Bini et. al. 1997 *Int. J. Mod. Phys. D* **6** 143-98).

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1. Introduction

Inertial forces, such as centrifugal and Coriolis forces, have proven to be helpful in Newtonian mechanics. Quite a lot of attention has been given to generalizing the concept to General Relativity. In fact the last fifteen years there has been a hundred or so papers related to inertial forces in General Relativity. For an overview see [1].

Many of these articles are related to particular types of spacetimes, and special types of motion. There are also a few that are completely general. This article is of the

latter kind. The scope is to develop a covariant formalism, applicable to any spacetime, and any motion of a test particle, using an arbitrary reference congruence of timelike worldlines. In view of the already existing bulk of papers we will keep the introductory remarks to a minimum here and just outline the contents of the article.

In section 2 we introduce the basic notation of the article.

In section 3 we derive a spatial curvature measure for a spacetime trajectory. We do this by projecting the trajectory down along the reference congruence onto the local time slice. We also derive how the time derivative of the speed relative to the congruence is related to the four-acceleration of the test particle. The resulting equations we put together to form a single equation that relates the test particle four-acceleration (and four-velocity) to the spatial curvature, the time derivative of the speed and the local derivatives of the congruence four-velocity. The terms connected to the congruence derivatives can be regarded as inertial forces. We also express the four-acceleration of the particle in terms of the experiences comoving forces, as well as in terms of the forces as *given* by the congruence observers.

In section 4 we introduce a different kind of spatial curvature measure. The new curvature measure is such that when we are following a straight line with respect to this measure, the spatial distance traveled (as defined by the congruence) is minimized (with respect to variations in the spatial curvature). This is in fact not the case for the standard projected curvature when the congruence is shearing. Using the new curvature measure we create a slightly different inertial force formalism.

In section 5 we consider general conformal rescalings of spacetime, and how these affect the inertial force formalism.

In section 6 we consider a foliation of spacetime into spacelike time slices and a corresponding orthogonal congruence. Given a labeling t of the time slices we rescale away time dilation with respect to t. Relating spatial curvature etc to the rescaled spacetime, but considering the real (non-rescaled) forces, we find an inertial force formalism that is very similar to the already derived formalisms of this paper. We show that a geodesic photon always follows a straight line in the sense of section 4. We also show that it follows a straight line in the projected sense if the congruence is shearfree. These results allows certain generalizations of the optical geometry (for an introduction to optical geometry see e.g. [2]) as will be pursued in a companion paper [3].

In section 7 we show that the fact that a geodesic photon follows a straight line in the new sense relative to the rescaled spacetime follows from Fermat's principle.

In section 8 we introduce two new curvature measures related to geodesic photons, and what we *see* as straight, and use these in the inertial force formalism.

In section 9 we summarize the inertial force formalisms (excepting those related to rescalings) connected to the various introduced curvature measures.

In section 10 we rewrite the four-covariant formalism as a three-dimensional formalism, for the particular case of vanishing shear (assuming only isotropic expansion). While fully relativistically correct, in this form the inertial force formalism is very similar

to its Newtonian counterpart.

In section 11 we derive a spacetime transport law of a vector, corresponding to *spatial* parallel transport with respect to the spatial geometry defined by the reference congruence.

In section 12 we consider an alternative approach to inertial forces resting on the transport equation of section 11.

In section 13 we connect to the approach of Jantzen et. al.

In section 14 we conclude the article. Then follows the appendixes.

2. The basic notation

In a general spacetime, we consider an arbitrary reference congruence of timelike worldlines of four-velocity η^μ. Each such worldline corresponds to events at a single spatial point in our frame of reference. We can split the four velocity v^μ of a test particle into a part parallel to η^μ and a part orthogonal to η^μ:

$$v^\mu = \gamma(\eta^\mu + vt^\mu). \tag{1}$$

Here v is the speed of the test particle relative to the congruence and γ is the corresponding γ-factor. The vector t^μ is a normalized spatial vector (henceforth vectors that are orthogonal to η^μ will be referred to as *spatial* vectors), pointing in the (spatial) direction of motion.

Projected spatial curvature and direction of curvature we will denote by R and n^μ, the latter being a normalized spatial vector. By projected curvature we mean that we project the spacetime trajectory in question down along the congruence onto the local slice[1] and evaluate the spatial curvature there. There are also several alternative definitions of curvature and curvature direction. In particular we will us $\bar{R}$ and $\bar{n}^\mu$ to denote what we will call the 'new-straight' curvature and curvature direction, to be introduced in section 4.

Throughout the article we will use $c = 1$ and adopt the spatial sign convention $(-, +, +, +)$. The projection operator[2] along the congruence then takes the form $P^\alpha{}_\beta \equiv g^\alpha{}_\beta + \eta^\alpha\eta_\beta$. We also find it convenient to introduce the suffix $\perp$. When applied to a four-vector, as in $[K^\mu]_\perp$, it selects the part within the brackets that is perpendicular to both η^μ and t^μ.

3. Inertial forces using the projected curvature

The objective with this section is to go from the spacetime equations of motion for a test particle and derive an expression for R, n^μ and the time derivative of v, in terms

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[1]If the congruence has no rotation there exists a finite sized slicing orthogonal to the congruence. If the congruence is rotating we can still introduce a slicing that is orthogonal at the point in question. It is easy to realize that whatever such locally orthogonal slicing we choose, the projected curvature and curvature directions will be the same, and are thus well defined.

[2]Applying this tensor to a vector extracts the spatial (i.e. orthogonal to η^μ) part of the vector.

of v and t^μ for given forces and congruence behavior.

3.1. The projected curvature and curvature direction

The idea behind the projected curvature with respect to the congruence is illustrated by figure 1. Notice that the time-slice we are depicting is only assumed to be orthogonal to the congruence at the point where the test particle worldline intersects the slice.
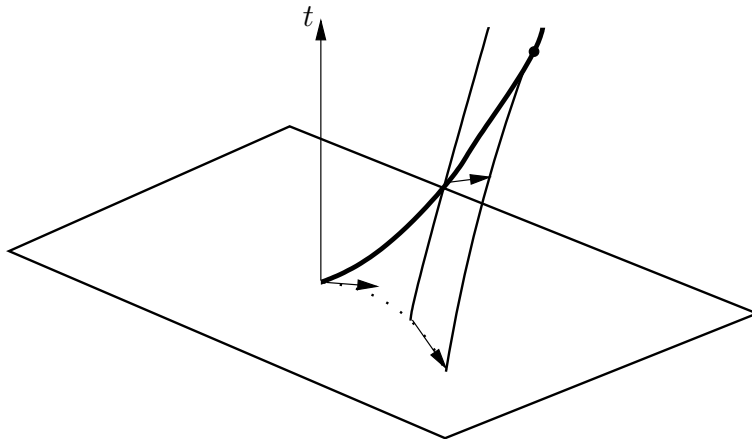

Figure 1. A 2+1 illustration of a projection of a spacetime trajectory onto a time slice, seen from freely falling coordinates, locally comoving with the reference congruence.

Taking the covariant derivative $\frac{D}{D\tau}$ along the test particle worldline, of (1) we readily find

$$\left[\frac{Dv^\mu}{D\tau}\right]_\perp = \gamma^2[a^\mu]_\perp + \gamma^2 v[t^\alpha\nabla_\alpha\eta^\mu]_\perp + \gamma v\left[\frac{Dt^\mu}{D\tau}\right]_\perp. \tag{2}$$

Here a^μ is the four-acceleration of the congruence. Now we want to relate the covariant derivative of t^μ in (2) to the projected curvature. As concerns the $\perp$-part of this we can consider the covariant derivative to stem from a two-step process. First we transport it along the curved projected trajectory, then we Lie-transport it up along the congruence as depicted in figure 1[3].

Alternatively, we may in the style of [4], consider the worldsheet spanned by the congruence lines that are crossed by the test particle worldline. On this sheet we can uniquely extend the forward vector t^μ, defined along the test particle worldline, into a vector field that is tangent to the sheet, normalized and orthogonal to η^μ. Considering an arbitrary smooth extension of this field t^μ around the sheet, the projected curvature can be written as $\frac{n^\mu}{R} = [t^\alpha\nabla_\alpha t^\mu]_\perp$. We also realize that, as concerns the $\perp$-part, this field will be Lie-transported into itself (in the η^μ direction). Thus we have $[\eta^\alpha\nabla_\alpha t^\mu]_\perp = [t^\alpha\nabla_\alpha\eta^\mu]_\perp$. Then we we can write

$$\left[\frac{Dt^\mu}{D\tau}\right]_\perp = [\gamma(\eta^\alpha + vt^\alpha)\nabla_\alpha t^\mu]_\perp \tag{3}$$

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[3]Letting $ds = vd\tau_0 = v\gamma d\tau$, we have in freely falling coordinates $[dt^\mu]_\perp = \frac{n^\mu}{R}ds + [t^\alpha\nabla_\alpha\eta^\alpha]_\perp d\tau_0$ from which (4) follows immediately.

$$= \gamma [t^\alpha \nabla_\alpha \eta^\mu]_\perp + \gamma v \frac{n^\mu}{R} \tag{4}$$

Using this together with (2) we get

$$\frac{1}{\gamma^2} \left[\frac{Dv^\mu}{D\tau} \right]_\perp = [a^\mu]_\perp + 2v [t^\alpha \nabla_\alpha \eta^\mu]_\perp + v^2 \frac{n^\mu}{R}. \tag{5}$$

So here is a general contravariant expression for the local projected curvature of a spacetime trajectory.

3.2. The speed change per unit time

Now we would like a corresponding expression for the speed change per unit time. We have $\gamma = -v^\alpha \eta_\alpha$. Differentiating both sides of this expression with respect to the proper time τ along the trajectory readily yields

$$\gamma^3 v \frac{dv}{d\tau} = -\frac{Dv^\alpha}{D\tau} \eta_\alpha - \frac{D\eta^\alpha}{D\tau} v_\alpha \tag{6}$$

$$= -\frac{Dv^\alpha}{D\tau} \left(\frac{v_\alpha}{\gamma} - vt_\alpha \right) - \frac{D\eta^\alpha}{D\tau} \gamma (\eta_\alpha + vt_\alpha) \tag{7}$$

$$= vt_\alpha \frac{Dv^\alpha}{D\tau} - \frac{D\eta^\alpha}{D\tau} \gamma vt_\alpha. \tag{8}$$

In the last equality we used the normalization of η^μ and v^μ. Notice that the differentiation is along the trajectory in question, so we have

$$\frac{D\eta^\alpha}{D\tau} = \gamma (\eta^\rho + vt^\rho) \nabla_\rho \eta^\alpha. \tag{9}$$

Using this in (8) we readily find

$$\frac{1}{\gamma^2} \frac{Dv^\alpha}{D\tau} t_\alpha = t_\alpha (\eta^\rho + vt^\rho) \nabla_\rho \eta^\alpha + \gamma \frac{dv}{d\tau}. \tag{10}$$

So here we have a covariant equation for the speed change as well.

3.3. Putting it together

Multiplying (10) by t^μ and adding it to (5), we get a single vector equation that relates the four-acceleration to both the speed change and the projected spatial curvature

$$\frac{1}{\gamma^2} \left(\left[\frac{Dv^\mu}{D\tau} \right]_\perp + \frac{Dv^\alpha}{D\tau} t_\alpha t^\mu \right) = [a^\mu]_\perp + 2v [t^\alpha \nabla_\alpha \eta^\mu]_\perp + v^2 \frac{n^\mu}{R} +$$
$$t^\mu t_\alpha (\eta^\rho + vt^\rho) \nabla_\rho \eta^\alpha + \gamma \frac{dv}{d\tau} t^\mu. \tag{11}$$

This can be simplified to

$$\frac{1}{\gamma^2} P^\mu{}_\alpha \frac{Dv^\alpha}{D\tau} = a^\mu + 2v [t^\alpha \nabla_\alpha \eta^\mu]_\perp + vt^\mu t^\alpha t^\rho \nabla_\rho \eta_\alpha + \gamma \frac{dv}{d\tau} t^\mu + v^2 \frac{n^\mu}{R}. \tag{12}$$

So here we have a generally covariant relation between the four-acceleration, the projected curvature and the speed change.

3.4. Experienced forces and the kinematical invariants

To make it more clear what an observer performing the specified motion actually experiences, we can rewrite the left hand side of (12) in terms of the experienced forward thrust[4] $F_{\parallel}$ and the experienced sideways thrust $F_{\perp}$. This is a simple exercise of special relativity performed in Appendix B. We may then write

$$\frac{1}{m\gamma^2}\left(\gamma F_{\parallel}t^{\mu} + F_{\perp}m^{\mu}\right) = a^{\mu} + 2v\left[t^{\alpha}\nabla_{\alpha}\eta^{\mu}\right]_{\perp} + vt^{\mu}t^{\alpha}t^{\rho}\nabla_{\rho}\eta_{\alpha} \tag{13}$$

$$+ \gamma\frac{dv}{d\tau}t^{\mu} + v^2\frac{n^{\mu}}{R}.$$

Here m^{μ} is a normalized vector perpendicular to t^{μ} and η^{μ}. We may alternatively express (13) in terms of the kinematical invariants of the congruence, defined in Appendix A. From the definitions follows [5][5]

$$\nabla_{\nu}\eta_{\mu} = \omega_{\mu\nu} + \theta_{\mu\nu} - a_{\mu}\eta_{\nu}. \tag{14}$$

We then readily find

$$\frac{1}{m\gamma^2}\left(\gamma F_{\parallel}t^{\mu} + F_{\perp}m^{\mu}\right) = a^{\mu} + 2v\left[t^{\beta}(\omega^{\mu}{}_{\beta} + \theta^{\mu}{}_{\beta})\right]_{\perp} + vt^{\alpha}t^{\beta}\theta_{\alpha\beta}t^{\mu} \tag{15}$$

$$+ \gamma\frac{dv}{d\tau}t^{\mu} + v^2\frac{n^{\mu}}{R}.$$

Here we have a covariant expression for the relation between spatial projected curvature and the speed change per unit time in terms of the experienced forces, given the kinematical invariants of the congruence.

3.5. Forces as experienced by the congruence observers

It may also be interesting to know what forces are needed to be given, by the observers following the congruence, in order to keep the test particle on the path in question. This again is a simple exercise of special relativity carried out in Appendix C where we readily show that

$$\frac{1}{\gamma^2}P^{\mu}{}_{\alpha}\frac{Dv^{\alpha}}{D\tau} = \frac{1}{\gamma m}(F_{c\parallel}t^{\mu} + F_{c\perp}m^{\mu}). \tag{16}$$

Here $F_{c\parallel}$ and $F_{c\perp}$ are the experienced given forces parallel and perpendicular to the direction of motion. When expressing the forces as given by the congruence observers, it seems reasonable to express the velocity change relative to local congruence time $d\tau_0$, given simply by $d\tau_0 = \gamma d\tau$. Then (15) takes the form

$$\frac{1}{m\gamma}\left(F_{c\parallel}t^{\mu} + F_{c\perp}m^{\mu}\right) = a^{\mu} + 2v\left[t^{\beta}(\omega^{\mu}{}_{\beta} + \theta^{\mu}{}_{\beta})\right]_{\perp} + vt^{\alpha}t^{\beta}\theta_{\alpha\beta}t^{\mu} \tag{17}$$

$$+ \gamma^2\frac{dv}{d\tau_0}t^{\mu} + v^2\frac{n^{\mu}}{R}.$$

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[4]By definition the observers forward direction is the direction from which he sees the congruence points coming (assuming he has some way of seeing them).

[5]Note that the sign of $\omega_{\mu\nu}$ is a matter of convention.

Here we have thus the inertial force equation explicitly in terms of the given forces. As a simple application we may consider a rotating merry-go-round with a railway track running straight out from the center. Suppose that we let a railway wagon move with constant speed along the track. Then (17) gives us the forces on the railway track[6].

3.6. Discussion

Looking back at (15) and (17), it is easy to put names to the various terms. On the left hand side we have the real experienced forces, as received and given respectively, in the forward and sidewards direction. On the right hand side we have first three terms that we may call inertial forces[7] given that we multiply them by $-m$:

$$\text{Acceleration}: -ma^{\mu} \tag{18}$$

$$\text{Coriolis}: \quad -2mv\left[t^{\beta}(\omega^{\mu}{}_{\beta} + \theta^{\mu}{}_{\beta})\right]_{\perp} \tag{19}$$

$$\text{Expansion}: \quad -mvt^{\alpha}t^{\beta}\theta_{\alpha\beta}t^{\mu}. \tag{20}$$

From a Newtonian point of view we may be tempted to call the first inertial force 'Gravity' rather than 'Acceleration'. On the other hand, for the particular case of using points fixed on a a rotating merry-go-round as reference congruence, the term would correspond to what we normally call centrifugal force. To avoid confusion we simply label this term 'Acceleration'. As regards the second term the naming is quite obvious[8]. The third term is non-zero if the reference grid is expanding or contracting in the direction of motion. For positive $t^{\alpha}t^{\beta}\theta_{\alpha\beta}$ the term has the form of a viscous damping force although for negative $t^{\alpha}t^{\beta}\theta_{\alpha\beta}$ it is rather a velocity proportional driving force. The existence of this term illustrates (for instance) that if we are using an expanding reference frame, a real force in the direction of motion is needed to keep the velocity relative to the reference frame fixed.

The two last terms of (15) and (17) are

$$\gamma^2 \frac{dv}{d\tau_0} t^{\mu} + v^2 \frac{n^{\mu}}{R}. \tag{21}$$

These we do not regard as inertial forces, but rather as descriptions of the motion (acceleration) relative to the reference frame. Notice that the formalism is well defined for arbitrary spacetimes and arbitrary timelike congruence lines.

3.7. A note on alternative interpretations

Quite commonly the term that we are here denote 'Expansion', is included with the $dv/d\tau_0$-term (multiplied by $-m$) and these two terms are collectively denoted the 'Euler'

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[6]After we have calculated $\omega^{\mu}{}_{\beta}$ and a^{μ} (for this case $\theta^{\mu}{}_{\beta} = 0$). See section 10.2.

[7]Actually exactly what we denote inertial force is subjective to a degree. For instance we could multiply all terms in (15) and (17) by γ and define the inertial forces accordingly.

[8]As can be seen from (14) (multiplied by t^{β}), the momentary velocities of the congruence points (relative to an inertial system momentarily comoving with the congruence) in the direction of motion is determined by $t^{\beta}(\omega^{\mu}{}_{\beta} + \theta^{\mu}{}_{\beta})$. Selecting the perpendicular part gives a measure of the sideways perpendicular velocities of the reference frame, naturally related to Coriolis.

force. This hides (or at least makes less manifest) the above mentioned feature that a real force is needed to keep a fixed velocity relative to an expanding reference frame. Indeed this lack was one of the original inspirations for making this paper. In section 12 we present an alternative approach to inertial forces resting on the notion of spatial parallel transport (of the relativistic three-momentum relative to the congruence). Then the expansion term arises naturally if we are using a norm-preserving law of spatial parallel transport.

Also, quite commonly the last term, when multiplied by -m, in (15) and and (17) respectively, is denoted the centrifugal force. This notation is however not matching the standard definition, where the centrifugal force comes from the acceleration due to the rotation of the reference frame rather than from the motion of the particle relative to the reference frame. See appendix F for further discussion of this.

If one interprets (like in e.g. [6]) the two terms related to accelerations relative to the reference frame (when multiplied by $-m$) as inertial forces – the whole equation takes a form of a balance equation between inertial forces. As interpreted in this article however, the inertial force equation is of the standard type $F_{\text{real}} + F_{\text{inertial}} = ma_{\text{relative}}$, where the acceleration relative to the reference frame corresponds to the last two terms of (15) and (17).

In Appendix F, we briefly review inertial forces in Newtonian mechanics and show that the derived formalism (and interpretation) of this paper is conforming (as far as that is possible) with the standard Newtonian formalism of inertial forces, in the limit of small velocities. We also discuss the possibility to view the terms related to the relative acceleration as inertial forces. For further understanding of the viewpoint that the last two terms are mere descriptions of the motion (acceleration) relative to the reference frame, see also section 12.

4. A different type of curvature radius

In the preceding section we used somewhat different techniques in deriving the perpendicular and the parallel parts. One might argue that the derivation of the perpendicular part, i.e. the curvature, was in a sense less local than the derivation of the forward part.

The heart of the matter lies in exactly where one measures spatial distances. In figure 2 we illustrate the difference between the on-slice distance $d\bar{s}$ and the at-trajectory distance ds.

To gain some intuition, we consider a finite slice, orthogonal to the congruence at the point where the test particle worldline intersects the slice, and a projection of the worldline down along the congruence onto the slice. The curvature radius, as defined in the preceding section, is such that when it is infinite, the *on-slice* distance is locally minimized[9]. Perhaps it would be more natural however to define a curvature radius

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[9]Strictly speaking it is necessary for the projected curvature to vanish at the point in question in order for the projected trajectory to minimize the distance on the slice. Note however that the
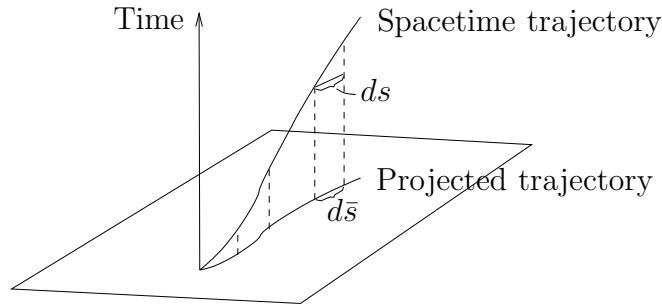
Figure 2. The difference between at-trajectory distance ds and on-slice distance $d\bar{s}$.

such that when it is infinite, the *at-trajectory* distance is minimized. Obviously these two definitions will coincide if there for instance is a Killing symmetry, and we adapt the congruence to the Killing field. For this case $ds = d\bar{s}$, and the two curvature measures will coincide. But in general it is perhaps not so obvious that they will, and indeed we will find that they do not coincide in general.

4.1. Defining a straight line via a variational principle

We would now like to introduce a new notion of straight trajectories, as those that minimize the integrated ds. We may formulate the problem of finding trajectories that are straight in the new sense via a variational principle. We thus introduce an action, for an arbitrary spacetime trajectory $x^{\mu}(\lambda)$, connecting two fixed spacetime points

$$\Delta s = \int \sqrt{P_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}} d\lambda. \tag{22}$$

We define a corresponding Lagrangian as

$$L = \sqrt{P_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}}. \tag{23}$$

Now we are interested in how a variation $x^{\mu}(\lambda) \rightarrow x^{\mu}(\lambda) + \delta x^{\mu}(\lambda)$ affects the action. Analogous (precisely) to the derivation of the Euler Lagrange equations we find (to first order in the change δx^{μ})

$$\delta \Delta s = \int \left[\frac{\partial L}{\partial x^{\mu}} - \frac{d}{d\lambda} \frac{\partial L}{\partial \frac{dx^{\mu}}{d\lambda}} \right] \delta x^{\mu} d\lambda. \tag{24}$$

This expression holds whatever parameterization we choose. In particular choosing the integrated local distance s itself as parameter, the Lagrangian function is unit[10] along the trajectory. For this choice of parameter, expanding (24) using (23) is particularly

projected curvature, as defined in the previous section, along the test particle worldline will not in general coincide with the spatial curvature of the corresponding point along the projected trajectory (except at the point of intersection).

[10]So $L = 1$, meaning that the absolute derivatives of L vanishes, whereas in general the partial derivatives do not.

simple, and the result is

$$\delta\Delta s = \frac{1}{2}\int\left[\frac{\partial P_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - 2\frac{d}{ds}\left(P_{\mu\beta}\frac{dx^\beta}{ds}\right)\right]\delta x^\mu ds \tag{25}$$

$$= \frac{1}{2}\int\left[(\partial_\mu P_{\alpha\beta})\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - 2(\partial_\rho P_{\mu\beta})\frac{dx^\rho}{ds}\frac{dx^\beta}{ds} - 2P_{\mu\beta}\frac{d^2 x^\beta}{ds^2}\right]\delta x^\mu ds \tag{26}$$

Also, using $\frac{d\tau}{ds} = \frac{1}{\gamma v}$, it is easy to prove that

$$\frac{d^2 x^\beta}{ds^2} = \frac{1}{\gamma v}\frac{d}{d\tau}\left(\frac{1}{\gamma v}\frac{dx^\beta}{d\tau}\right) = ... = -\frac{1}{v^3}\frac{dv}{d\tau}\frac{dx^\beta}{d\tau} + \frac{1}{\gamma^2 v^2}\frac{d^2 x^\beta}{d\tau^2}. \tag{27}$$

Inserting this into (26) we get

$$\delta\Delta s = \frac{1}{2}\int\left[(\partial_\mu P_{\alpha\beta})\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - 2(\partial_\rho P_{\mu\beta})\frac{dx^\rho}{ds}\frac{dx^\beta}{ds} - 2P_{\mu\beta}\frac{1}{\gamma^2 v^2}\frac{d^2 x^\beta}{d\tau^2}\right.$$
$$\left. +2P_{\mu\beta}\frac{1}{v^3}\frac{dv}{d\tau}\frac{dx^\beta}{d\tau}\right]\delta x^\mu ds. \tag{28}$$

Notice that while not explicitly covariant, this expression holds (to first order) whatever coordinates we use[11]. In particular it holds using locally inertial coordinates. We can therefore change all ordinary derivatives above to their covariant analogue [12]

$$\delta\Delta s = \frac{1}{2}\int\left[(\nabla_\mu P_{\alpha\beta})\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - 2(\nabla_\rho P_{\mu\beta})\frac{dx^\rho}{ds}\frac{dx^\beta}{ds} - 2P_{\mu\beta}\frac{1}{\gamma^2 v^2}\frac{D^2 x^\beta}{D\tau^2}\right.$$
$$\left. +2P_{\mu\beta}\frac{1}{v^3}\frac{dv}{d\tau}\frac{dx^\beta}{d\tau}\right]\delta x^\mu ds. \tag{29}$$

Now we would like to see how this expression depends on R. The inertial force formula (12) can be written as

$$\frac{1}{\gamma^2}P_{\mu\beta}\frac{D^2 x^\beta}{D\tau^2} = \eta^\alpha\nabla_\alpha\eta_\mu + v\left(2t^\alpha\nabla_\alpha\eta_\mu - t_\mu t^\alpha t^\beta\nabla_\alpha\eta_\beta\right) \tag{30}$$
$$+ \gamma^2\frac{dv}{d\tau_0}t_\mu + v^2\frac{n_\mu}{R}.$$

Using this in (29), together with $\frac{dx^\alpha}{ds} = \frac{1}{v}(\eta^\alpha + vt^\alpha)$, we find after simplification

$$\delta\Delta s = -\int\frac{1}{v}\left[v\frac{n_\mu}{R} + t^\beta\nabla_\mu\eta_\beta + \eta_\mu t^\beta a_\beta - t_\mu t^\alpha t^\beta\nabla_\alpha\eta_\beta\right. \tag{31}$$
$$\left. +v\eta_\mu t^\rho t^\beta\nabla_\rho\eta_\beta + t^\beta\nabla_\beta\eta_\mu\right]\delta x^\mu ds.$$

This expression we may now simplify a bit. From (14) (using the antisymmetry of $\omega_{\mu\nu}$) readily follows

$$2t^\beta\theta_{\beta\mu} = t^\beta(\nabla_\mu\eta_\beta + \nabla_\beta\eta_\mu) + \eta_\mu t^\beta a_\beta. \tag{32}$$

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[11]This is evident since the original equation (22), and the derivation thus far, holds for arbitrary coordinates.

[12]Having done this, we may as a check-up insert the explicit expressions for the covariant derivatives, with the affine connection, and see that the affine connection terms indeed cancel out.

Using this in (31), also using $t^\beta \theta_{\beta\mu} = [t^\beta \theta_{\beta\mu}]_\perp + t_\mu t^\alpha t^\beta \nabla_\alpha \eta_\beta$, the expression within the brackets of (31) is readily decomposed into an η^μ-part a t^μ-part and a part that is perpendicular to both t^μ and $\eta^{\mu 13}$

$$\delta\Delta s = -\int \frac{1}{v}\left[v\frac{n_\mu}{R} + 2[t^\alpha \theta_{\alpha\mu}]_\perp + \eta_\mu v t^\alpha t^\beta \nabla_\alpha \eta_\beta + t_\mu t^\alpha t^\beta \nabla_\alpha \eta_\beta \right]\delta x^\mu ds. \quad (33)$$

Now, for the spacetime trajectory to be a solution to the optimization problem, allowing for arbitrary variations δx^μ, the expression within the brackets must vanish. Studying the η^μ and t^μ parts yields

$$t^\alpha t^\beta \nabla_\alpha \eta_\beta = 0. \quad (34)$$

What this means is that for a trajectory to optimize the integrated distance, the trajectory must never pass two close-lying congruence lines in a direction where there is expansion. This is actually quite natural since there is no penalty (increase of ds) in letting the spacetime trajectory follow a congruence line. To minimize the integrated ds, the spacetime trajectory must never cross between two infinitesimally displaced congruence lines unless there is a minimum distance separating them (implying zero expansion). It was to make this point clear that we didn't just use the Euler Lagrange equations directly before, but kept the expression for the change of the action under the variation.

We are however not really interested in minimizing the distance traveled with respect to the spacetime trajectory. In fact we just want to minimize the integrated distance with respect to the spatial curvature. Alternatively we could say that we want to solve the optimization problem with respect to variations perpendicular to t^μ and η^μ. We see immediately from (33) that this can be accomplished if we have

$$v\frac{n_\mu}{R} = -2[t^\alpha \theta_{\alpha\mu}]_\perp. \quad (35)$$

We thus find that in general when there is a non-zero expansion-shear tensor, the new sense of straightness differs from the projected version. If we have a Killing symmetry, and adapt the congruence to the Killing field, the congruence will necessarily be rigid and thus $\theta_{\alpha\mu} = 0$. So for a congruence adapted to the Killing field the two curvature measures coincide, as anticipated. We also see that if we have isotropic expansion and no shear, so $\theta^\mu{}_\nu \propto \delta^\mu{}_\nu$, the new sense of straightness coincides with the projected version. This is also completely expected.

We may also notice that the projected curvature radius depends on the velocity as $1/R \propto 1/v$. The smaller the velocity the greater the spatial curvature (and thus the smaller the curvature *radius*). This feels quite natural, moving slowly between fixed congruence lines implies more time for expansion and shear effects to kick in, enabling greater detours (in the projected sense).

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[13]If trying to make sense, by simple thought experiments, of the various terms – keep in mind that while the integral of (33) corresponds to our original integral of (24), the *integrands* of these two equations are not in general the same (recall the partial integration undertaken in the derivation of Euler Lagrange's equations).

4.2. More intuition regarding the new-straight formalism

Assume that we have a diagonal $\theta_{\alpha\beta}$ (in inertial coordinates locally comoving with the congruence) where there is a lot of contraction in, say the x-direction, and no expansion or contraction in the y-direction. Consider then, in a 2+1 spacetime, the problem of minimizing the integrated distance while connecting the opposing corners of a spacetime box as illustrated in figure 3.
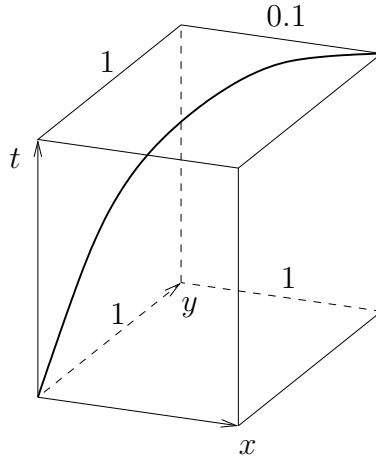

Figure 3. A box spanned by the generating congruence, seen in coordinates adapted to these. The proper x and y distances of the sides of the box are indicated. At large t, proper distances in the x-direction are smaller than at small t. We understand that a trajectory minimizing the integrated proper distance will in fact curve (relative to the coordinates in question), as illustrated by the thick line.

If there is severe contraction in the x-direction it will towards the end of the trajectory be very cheap to travel in the x-direction. Thus the trajectory should initially start along the y-axis before turning back and at the end almost follow the x-axis. Thus we have some intuitive understanding for why a trajectory that is straight in the new sense has a projected curvature (in general)[14].

4.3. Defining a new curvature measure $\bar{R}$

Now we know what kind of motion that is straight in the new sense. Notice that the projected curvature radius of such a line depends on both the spatial direction t^μ and the velocity v. For any given t^μ and v we can however define a new curvature radius and a direction of curvature, for a general trajectory of the t^μ and v in question, by how fast and in what direction the trajectory deviates from a corresponding line that is straight in the new sense. See figure 4.

Let n_0^k and R_0 denote projected curvature direction and radius respectively for the projection of a trajectory that is straight in the new sense, and let ds denote proper

[14]Strictly speaking we may at least understand that such a trajectory (as depicted in figure 3) *can* be shorter than a coordinate straight trajectory connecting the opposite corners of the box.
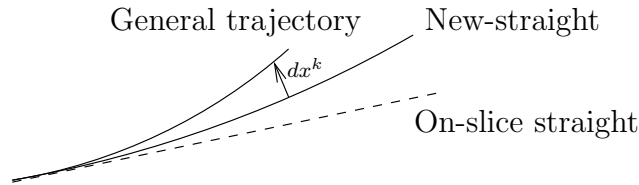
Figure 4. A projection onto the local slice of a new-straight trajectory, of a certain t^μ and v, and a corresponding projection of another trajectory of the same t^μ and v. The deviation between the lines can be used to define a new curvature and curvature direction. We may understand that to lowest non-zero order in time, the on-slice deviation is the same as the at trajectory deviation (thinking in 2+1 spacetime). So the projected deviation should do nicely as a measure also of the at-trajectory deviation.

distance along a curve. To lowest non-zero order the deviation is given by

$$dx^k = \frac{n^k}{R}\frac{ds^2}{2} - \frac{n_0^k}{R_0}\frac{ds^2}{2}. \tag{36}$$

Letting a bar denote curvature direction and curvature in the new sense, we define (see figure 4) $dx^k = \frac{\bar{n}^k}{R}\frac{ds^2}{2}$. Using this together with (35) for n_0^k and R_0 in (36) we readily find

$$\frac{\bar{n}_\mu}{\bar{R}} = \frac{n_\mu}{R} + \frac{2}{v}[t^\beta \theta_{\beta\mu}]_\perp. \tag{37}$$

So here is the new curvature measure in terms of the projected curvature.

4.4. The inertial force formalism using the new curvature measure

Using (37) in the inertial force equation (15), we immediately find the corresponding equation for the new curvature measure

$$\frac{1}{m\gamma^2}P^\mu{}_\alpha\frac{Dv^\alpha}{D\tau} = a^\mu + 2vt^\beta\omega^\mu{}_\beta + vt^\alpha t^\beta\theta_{\alpha\beta}t^\mu + \gamma^2\frac{dv}{d\tau_0}t^\mu + v^2\frac{\bar{n}^\mu}{\bar{R}}. \tag{38}$$

We see that the inertial force expression in fact is a bit cleaner with the new representation of curvature. The difference lies in the Coriolis term, the second term on the right hand side, which contains no shear-expansion term now.

Notice that while we introduced the concept of curvature in the new sense easily enough, it is a bit more abstract than the projected curvature which can be defined via a projection onto a single locally well defined spatial geometry. It would appear that no such geometry applies to the new sense of curvature[15]. For every fixed speed v, we know however what is straight in every direction, and that is sufficient to define a curvature.

[15]The argument goes like this. Suppose that we have *some* spatial geometry on the local slice such that trajectories that are straight in the new sense, and of a certain v, when projected down along η^μ are straight relative to the spatial geometry. Consider then trajectories with a different velocity v, that are also straight in the new sense. These will (assuming a non-zero $[t^\beta\theta_{\beta\mu}]_\perp$) according to (35) have another projected curvature (relative to the standard spatial geometry). They will thus deviate (to second order) from the corresponding projected trajectories of the previous velocity. Thus these cannot also be straight relative to the spatial geometry in question.

We could in principle consider projecting along some other local congruence down to a local slice

Certainly the new definition of curvature is in some sense more 'local' than the projected one. It feels like a better match with the forward part (connected to $dv/d\tau$) now.

4.5. A joint expression

For brevity it will prove useful to have a single expression that incorporates both the projected and the new-straight formalisms. We therefore let the suffix 's' denote either 'ps' standing for projected straight, or 'ns' standing for new-straight. Introducing $C_{\text{ps}} = 1$, $C_{\text{ns}} = 0$ we can then write:

$$\frac{1}{m\gamma^2}\left(\gamma F_{\parallel} t^{\mu} + F_{\perp} m^{\mu}\right) = a^{\mu} + 2v\left[t^{\beta}(\omega^{\mu}{}_{\beta} + C_{\text{s}}\theta^{\mu}{}_{\beta})\right]_{\perp} + vt^{\alpha}t^{\beta}\theta_{\alpha\beta}t^{\mu} \qquad (39)$$

$$+ \gamma^2 \frac{dv}{d\tau_0}t^{\mu} + v^2 \frac{n_{\text{s}}^{\mu}}{R_{\text{s}}}.$$

Here $R_{\text{ps}} \equiv R$, $R_{\text{ns}} \equiv \bar{R}$ and analogously $n_{\text{ps}}^{\mu} \equiv n^{\mu}$ and $n_{\text{ns}}^{\mu} \equiv \bar{n}^{\mu}$.

4.6. A comment on another alternative

A line that is spatially straight in the new sense is such that the distance taken relative to the congruence is minimized (with respect to variations in the spatial curvature). One could alternatively consider optimizing the *arrival time* for a particle moving with a fixed speed, relative to the congruence, from one event (along some congruence line) to another congruence line. Considering for instance a static black hole (where there is time dilation) we understand that to optimize the arrival time it is beneficial to travel where there is relatively little time dilation (hence moving out and then back relative to a straight line). We may understand that a trajectory that is straight in the time-optimizing sense is curving inwards relative to a line that is straight in the projected sense. We will not pursue the issue further here, but we will comment on it again in section 7.

5. General conformal rescalings

In a series of papers Abramowicz et. al. (see e.g [2, 7, 8, 6, 9, 10]) investigated inertial forces in special and general cases using a certain conformally rescaled spacetime. In this section we consider how a general rescaling of the spacetime affects the inertial force formalism. In section 6 we will apply this formalism to the particular rescaling of Abramowicz et. al.

Study then an arbitrary rescaling of spacetime $\tilde{g}_{\mu\nu} = e^{-2\Phi}g_{\mu\nu}$. Relative to the rescaled geometry we can express the rescaled four-acceleration of the test particle in terms of the rescaled curvature, the rescaled rotation tensor etc. Letting a tilde on

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such that all the trajectories of a certain t^{μ} (but different v) that are straight in the new sense get the same projected trajectory. To have two effective congruences (to achieve the goal of a unique spatial geometry) seems, at least at first sight, quite contrived and we will not pursue the idea further here.

an object indicate that it is related to the rescaled spacetime, we just put tilde on everything in the joint expression (39) for both the projected and the new-straight formalisms. Notice that v and γ are unaffected by the rescaling[16] and we may omit the tilde on them

$$\frac{1}{\gamma^2}\tilde{P}^\mu{}_\alpha\frac{\tilde{D}^2 x^\alpha}{\tilde{D}\tilde{\tau}^2} = \tilde{a}^\mu + 2v\left[\tilde{t}^\beta(\tilde{\omega}^\mu{}_\beta + C_{\rm s}\tilde{\theta}^\mu{}_\beta)\right]_\perp + v\tilde{t}^\alpha\tilde{t}^\beta\tilde{\theta}_{\alpha\beta}\tilde{t}^\mu \tag{40}$$

$$+ \gamma^2\frac{dv}{d\tilde{\tau}_0}\tilde{t}^\mu + v^2\frac{\tilde{n}_{\rm s}^\mu}{\tilde{R}_{\rm s}}.$$

While we have rescaled the spacetime, we are still interested in knowing what a *real* observer experiences in terms of forward and sideways thrusts. Then we need to relate the four-acceleration relative to the rescaled spacetime to the four-acceleration of the non-rescaled spacetime. In Appendix D we show how the four-acceleration transforms under conformal transformations. The result is given by (D.14)

$$\frac{\tilde{D}^2 x^\mu}{\tilde{D}\tilde{\tau}^2} = e^{2\Phi}\frac{D^2 x^\mu}{D\tau^2} - \frac{dx^\mu}{d\tilde{\tau}}\frac{dx^\rho}{d\tilde{\tau}}\tilde{\nabla}_\rho\Phi - \tilde{g}^{\mu\rho}\tilde{\nabla}_\rho\Phi. \tag{41}$$

We know that $\frac{dx^\mu}{d\tilde{\tau}} = \gamma(\tilde{\eta}^\mu + v\tilde{t}^\mu)$ and $\frac{D^2 x^\mu}{D\tau^2} = \frac{1}{m}\left(\gamma F_\| t^\mu + F_\perp m^\mu\right)$, as derived in Appendix B. Also using $\tilde{t}^\mu = e^\Phi t^\mu$ and $\tilde{m}^\mu = e^\Phi m^\mu$, we can rewrite (41) as

$$\frac{1}{\gamma^2}\tilde{P}^\mu{}_\alpha\frac{\tilde{D}^2 x^\alpha}{\tilde{D}\tilde{\tau}^2} = \frac{1}{m\gamma^2}e^\Phi\left(\gamma F_\|\tilde{t}^\mu + F_\perp\tilde{m}^\mu\right) -$$

$$\left([\tilde{P}^{\mu\rho}\tilde{\nabla}_\rho\Phi]_\| + \frac{1}{\gamma^2}[\tilde{P}^{\mu\rho}\tilde{\nabla}_\rho\Phi]_\perp + v(\tilde{\eta}^\rho\tilde{\nabla}_\rho\Phi)\tilde{t}^\mu\right). \tag{42}$$

This we may now insert into (40) to get an expression for the real experienced forces, in terms of the motion relative to the rescaled spacetime, the rescaled expansion etc. The results follow immediately. Here is the rescaled version the inertial force expression

$$\frac{e^\Phi}{m\gamma^2}\left(\gamma F_\|\tilde{t}^\mu + F_\perp\tilde{m}^\mu\right) = \tilde{a}^\mu + [\tilde{P}^{\mu\rho}\tilde{\nabla}_\rho\Phi]_\| + \frac{1}{\gamma^2}[\tilde{P}^{\mu\rho}\tilde{\nabla}_\rho\Phi]_\perp + v(\tilde{\eta}^\rho\tilde{\nabla}_\rho\Phi)\tilde{t}^\mu$$

$$+2v\left[\tilde{t}^\beta(\tilde{\omega}^\mu{}_\beta + C_{\rm s}\tilde{\theta}^\mu{}_\beta)\right]_\perp + v\tilde{t}^\alpha\tilde{t}^\beta\tilde{\theta}_{\alpha\beta}\tilde{t}^\mu + \gamma^2\frac{dv}{d\tilde{\tau}_0}\tilde{t}^\mu + v^2\frac{\tilde{n}_{\rm s}^\mu}{\tilde{R}_{\rm s}}. \tag{43}$$

We notice that under general conformal rescalings, the inertial force formalism contains extra terms, making it more complicated in general.

6. Optical rescalings for a hypersurface-forming congruence

Now study the special case of a timelike hypersurface-forming congruence. The congruence must then obey $\omega_{\mu\nu} = 0$. Such a congruence can always be generated by introducing a foliation of spacetime specified by a single scalar function $t(x^\mu)$. We simply form $\eta_\mu = -e^\Phi\nabla_\mu t$ (recall that we are using the $(-,+,+,+)$-signature) where the scalar field Φ is chosen so that η^μ is normalized.

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[16]One can say that space is stretched as much as time, or that spacetime angles (and hence velocities) are preserved under a conformal rescaling.

Now consider a rescaling of spacetime by a factor $e^{-2\Phi}$. It follows that for displacements along the congruence we have $dt = d\tilde{\tau}$[17]. For this particular choice of Φ it is easy to prove, as is done in Appendix E, that $\tilde{a}^{\mu} = 0$. This is also easy to understand. The rescaling, apart from stretching space, removes time-dilation (lapse). Then it is obvious, from the point of view of maximizing proper time, that the congruence lines are geodesics in the rescaled spacetime. In the optically rescaled spacetime the congruence is still orthogonal to the same slices, hence $\tilde{\omega}^{\mu}{}_{\nu}$ vanishes.

6.1. The inertial force formalism in the rescaled spacetime

Before considering the effect of the rescaling, let us for comparison first have a look at the non-rescaled inertial force equation, for the congruence at hand. From (E.7) in Appendix E, we know that $a_{\mu} = P^{\alpha}{}_{\mu}\nabla_{\alpha}\Phi$. Using this together with $\omega_{\alpha\beta} = 0$ in (39), we are left with

$$\frac{1}{m\gamma^2}\left(\gamma F_{\|}t^{\mu} + F_{\perp}m^{\mu}\right) = P^{\mu\alpha}\nabla_{\alpha}\Phi + vt^{\alpha}t^{\beta}\theta_{\alpha\beta}t^{\mu} + C_{\mathrm{s}}2v\left[t^{\beta}\theta^{\mu}{}_{\beta}\right]_{\perp}$$
$$+ \gamma^2\frac{dv}{d\tau_0}t^{\mu} + v^2\frac{n_{\mathrm{s}}^{\mu}}{R_{\mathrm{s}}}. \tag{44}$$

Now consider a congruence and a conformal rescaling as described in the beginning of the section. Equation (43) is then simplified to

$$\frac{1}{m\gamma^2}e^{\Phi}\left(\gamma F_{\|}\tilde{t}^{\mu} + F_{\perp}\tilde{m}^{\mu}\right) = [\tilde{P}^{\mu\rho}\tilde{\nabla}_{\rho}\Phi]_{\|} + \frac{1}{\gamma^2}[\tilde{P}^{\mu\rho}\tilde{\nabla}_{\rho}\Phi]_{\perp} \tag{45}$$
$$+ v(\tilde{\eta}^{\rho}\tilde{\nabla}_{\rho}\Phi + \tilde{t}^{\alpha}\tilde{t}^{\beta}\tilde{\theta}_{\alpha\beta})\tilde{t}^{\mu} + C_{\mathrm{s}}2v\left[\tilde{t}^{\beta}\tilde{\theta}^{\mu}{}_{\beta}\right]_{\perp}$$
$$+ \gamma^2\frac{dv}{d\tilde{\tau}_0}\tilde{t}^{\mu} + v^2\frac{\tilde{n}_{\mathrm{s}}^{\mu}}{\tilde{R}_{\mathrm{s}}}.$$

While we loose the manifest connection to experienced four-acceleration, we can further simplify this by dividing the parallel parts of both sides by γ^2 yielding

$$\frac{1}{m\gamma^2}e^{\Phi}\left(\frac{F_{\|}}{\gamma}\tilde{t}^{\mu} + F_{\perp}\tilde{m}^{\mu}\right) = \frac{1}{\gamma^2}\tilde{P}^{\mu\rho}\tilde{\nabla}_{\rho}\Phi + \frac{v}{\gamma^2}(\tilde{\eta}^{\rho}\tilde{\nabla}_{\rho}\Phi + \tilde{t}^{\alpha}\tilde{t}^{\beta}\tilde{\theta}_{\alpha\beta})\tilde{t}^{\mu} \tag{46}$$
$$+ C_{\mathrm{s}}2v\left[\tilde{t}^{\beta}\tilde{\theta}^{\mu}{}_{\beta}\right]_{\perp} + \frac{dv}{d\tilde{\tau}_0}\tilde{t}^{\mu} + v^2\frac{\tilde{n}_{\mathrm{s}}^{\mu}}{\tilde{R}_{\mathrm{s}}}.$$

Comparing this inertial force equation with its non-rescaled analogue given by (44), we find that excepting tildes, γ-factors and a factor e^{Φ}, the only difference lies in the appearance of a $\tilde{\eta}^{\rho}\tilde{\nabla}_{\rho}\Phi = \frac{\partial\Phi}{\partial t}$-term within the expansion term. The occurrence of the extra term is quite natural considering that any time derivative in the spacetime rescaling will act as a spatial expansion. If we so wish, we can alternatively express (46) in terms of the non-rescaled kinematical invariants, see (D.6)-(D.10) (while still keeping the rescaled spatial curvature), thus effectively considering a rescaled *space* rather than a rescaled *spacetime*.

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[17]Contracting both sides of $\eta_{\mu} = -e^{\Phi}\nabla_{\mu}t$ by $\eta^{\mu} = \frac{dx^{\mu}}{d\tau}$ yields $1 = e^{\Phi}\frac{dx^{\mu}}{d\tau}\nabla_{\mu}t = e^{\Phi}\frac{dt}{d\tau}$. Using $d\tilde{\tau} = e^{-\Phi}d\tau$ we get $d\tilde{\tau} = dt$.

6.2. The projected curvature

As a particular example we consider motion along a line that is straight in the projected sense. The perpendicular part of (46) then becomes (recall that $C_{\mathrm{ps}} = 1$)

$$\frac{e^{\Phi}}{m} F_{\perp} \tilde{m}^{\mu} = \tilde{P}^{\mu\rho} \tilde{\nabla}_{\rho} \Phi + 2v\gamma^2 \left[\tilde{t}^{\beta} \tilde{\theta}^{\mu}{}_{\beta} \right]_{\perp}. \tag{47}$$

In particular, when the rescaled congruence is rigid (so $\tilde{\theta}_{\alpha\beta} = 0$), as in conformally static spacetimes (using a suitable rescaling and a corresponding congruence) the experienced comoving sideways force is independent of the velocity along the straight line. This is a well known result of optical geometry. Now we see also how this somewhat Newtonian flavor is broken (for the curvature measure at hand) in general when the rescaled shear-expansion tensor is non-zero.

6.2.1. Geodesic photons For geodesic particles the left hand side of (46) vanishes. In particular, for a geodesic photon, the forward part yields simply $\frac{dv}{d\tilde{\tau}_0} = 0$, and the perpendicular part yields simply

$$0 = 2v \left[\tilde{t}^{\beta} \tilde{\theta}^{\mu}{}_{\beta} \right]_{\perp} + v^2 \frac{\tilde{n}^{\mu}}{\tilde{R}}. \tag{48}$$

We see that the projected curvature vanishes for the free photon if we have

$$\tilde{\theta}^{\mu}{}_{\beta} \tilde{t}^{\beta} \propto \tilde{t}^{\mu}. \tag{49}$$

Knowing that $\tilde{\theta}^{\mu}{}_{\beta} = \tilde{\sigma}^{\mu}{}_{\beta} + \frac{\tilde{\theta}}{3} \tilde{P}^{\mu}{}_{\beta}$ we see that (49) is equivalent to $\tilde{\sigma}^{\mu}{}_{\beta} \tilde{t}^{\beta} \propto \tilde{t}^{\mu}$. We know that (may easily show that) $\tilde{\sigma}^{\mu}{}_{\beta} \tilde{\eta}^{\beta} = 0$. Knowing also that $\tilde{\sigma}_{\mu\nu}$ is a symmetric tensor it follows that in coordinates adapted to the congruence, only the spatial part of $\tilde{\sigma}_{\mu\nu}$ is nonzero. Also, for (49) to hold for arbitrary spatial directions $\tilde{t}^i$, we must have $\tilde{\sigma}^i{}_j \propto \delta^i{}_j$. Knowing also that the trace $\tilde{\sigma}^{\alpha}{}_{\alpha}$ always vanishes, it follows that $\tilde{\sigma}^{\mu}{}_{\nu}$ must vanish entirely. If a tensor vanishes in one system it vanishes in all systems. Thus we conclude that for photons to follow optical spatial geodesics in the (standard) projected meaning, the congruence must (relative to the rescaled space) be shearfree (and also rotationfree). It is not hard to show (see Appendix D) that we have $\tilde{\sigma}_{\mu\nu} = e^{-\Phi} \sigma_{\mu\nu}$ and $\tilde{\omega}_{\mu\nu} = e^{-\Phi} \omega_{\mu\nu}$. Thus also relative to the original spacetime geometry must the shear (and rotation) vanish. This result will be used in a companion paper [3] on generalizing the theory of optical geometry.

6.3. The new sense of curvature

As a particular example we consider motion along a line that is straight in the new sense. The perpendicular part of (46) then becomes

$$e^{\Phi} \frac{F_{\perp}}{m} \tilde{m}^{\mu} = \left[\tilde{P}^{\mu\rho} \tilde{\nabla}_{\rho} \Phi \right]_{\perp}. \tag{50}$$

Notice in particular the absence of γ factors in this expression. In a rescaled spacetime, with the new definition of curvature, the perpendicular part works just like in Newtonian gravity (up to a factor e^{Φ}). The experienced sideways force is independent of the velocity, even when the congruence is shearing (unlike when using the projected curvature).

6.3.1. Geodesic photons For geodesic particles the left hand side of (46) vanishes. In particular, for a geodesic photon, the forward part yields simply $\frac{dv}{d\tilde{\tau}_0} = 0$, and the perpendicular part yields simply

$$\frac{\tilde{\tilde{n}}^\mu}{\tilde{\tilde{R}}} = 0. \tag{51}$$

So a geodesic photon follows a line that is spatially straight in the new sense. This results will also be used in a forthcoming paper on generalizing the theory of optical geometry.

7. Fermat's principle and its connection to straightness in the new sense, in rescaled spacetimes

Fermat's principle (see [11] for a formal proof) tells us that a geodesic photon traveling from an event P to a nearby timelike trajectory Λ will do this in such a way that the time (as measured along Λ) is stationary[18]. In particular any null trajectory minimizing the arrival time is a geodesic.

By introducing any spacelike foliation of spacetime, and a corresponding future-increasing time coordinate t, optimizing the arrival time at Λ is equivalent to optimizing the coordinate time difference δt. In particular, assuming a hypersurface-forming generating congruence, we may introduce an orthogonal foliation and a corresponding time coordinate t. After rescaling the spacetime (to take away time dilation), coordinate time, velocity and spatial distance are related simply by $d\tilde{s} = v dt$. The total coordinate time δt needed for a particle (not necessarily a photon) moving with constant speed v from P to Λ can then be expressed as

$$\delta t = \int dt = \int \frac{1}{v} d\tilde{s} = \frac{1}{v} \int d\tilde{s}. \tag{52}$$

What this says is quite obvious: no time dilation and constant speed means that time is proportional to distance. In particular for a photon, having fixed speed $v = 1$, the coordinate time taken is minimized if and only if the integrated local (rescaled) distance is minimized. This in turn can hold only if the curvature in the new sense vanishes. So it in fact follows[19] from Fermat's principle that a geodesic photon[20] has zero curvature

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[18]By stationary we mean that it is a minimum or a saddle point with respect to variations in the set of all null trajectories connecting P to Λ. As an example we may consider a 2+1 spacetime where the spatial geometry is that of a sphere, and there is no time-dilation. Then a geodesic photon can take the long way around (following a great circle), rather than the short, in going from P to Λ. This would be a saddle point rather than a minimum.

[19]Strictly speaking, what we have shown is that *any* null geodesic that minimizes the arrival time, for the P and Λ in question, has vanishing $\tilde{\tilde{R}}$. It seems safe to assume that any sufficiently small (but finite) section of *any* null geodesic must correspond to minimizing the arrival time for *some* P and Λ (consider the equivalence principle). Since the argumentation holds for arbitrary P and Λ, it then follows that *any* null geodesic has vanishing $\tilde{\tilde{R}}$.

[20]Logically, we have here always referred to a photon geodesic with respect to the standard spacetime, which is precisely what we are after. We may however note that, for the particular spacetime

in the new sense relative to the optically rescaled spacetime. This is a verification of our earlier result (51) that was derived without reference to Fermat's principle.

Note also that in the rescaled spacetime any trajectory (not only null trajectories) of constant speed that is minimizing the spatial distance is also minimizing the arrival time. Hence the time-optimizing curvature measure as discussed briefly in section 4.6 is identical (up to a pure rescaling) to the new-straight curvature in the optically rescaled spacetime.

The connection between Fermat's principle, null geodesics and straight lines in the optical geometry was realized, for conformally static spacetimes, a long time ago. Now we see that with the new definition of curvature the connection holds in any spacetime.

8. Other photon related curvatures

Besides the already discussed curvature measures, and their relation to geodesic photons, it is not hard to come up with a couple of more approaches with different virtues and set-backs.

8.1. Curvature relative to that of a geodesic photon

The new sense of curvature has the virtue that, in the optically rescaled spacetime, geodesic photons follow spatially straight lines. On the other hand expressions like 'follow the photon' loose their meaning in the sense that two spacetime trajectories, cutting the same congruence lines and hence taking the 'same' spatial trajectory (as seen in coordinates adapted to the congruence) need not have the same measure of curvature.

We could try to keep the cake, while also eating it, by using a modified version of the projected curvature. We project the trajectory onto the local slice, but we define the curvature – not via the deviation from a straight line on the slice – but via the deviation from the projected trajectory of a geodesic photon. This definition of curvature can be applied without the restriction to a hypersurface-forming congruence. Also, regardless of rescalings photons will per definition follow straight lines. From (15) it immediately follows that a geodesic photon obeys

$$0 = [a^\mu]_\perp + 2\left[t^\beta(\omega^\mu{}_\beta + \theta^\mu{}_\beta)\right]_\perp + \frac{n^\mu}{R}. \tag{53}$$

We then introduce the new curvature as

$$\frac{n'^\mu}{R'} = \frac{n^\mu}{R} + [a^\mu]_\perp + 2\left[t^\beta(\omega^\mu{}_\beta + \theta^\mu{}_\beta)\right]_\perp. \tag{54}$$

transformation we are considering, there is no need to distinguish between null geodesics relative to the standard and the rescaled spacetime. A null worldline is a geodesic relative to the standard spacetime if and only if it is a geodesic relative to the rescaled spacetime. Indeed this follows from Fermat's principle (which in turn is very reasonable considering the equivalence principle) since neither null-ness, nor whether a null trajectory corresponds to a stationary arrival time or not, are affected by the conformal transformation. It can also readily be shown using (41) considering vanishing rescaled four-acceleration, evaluating $\frac{d^2\mathbf{x}}{dt^2}$ in originally freely falling coordinates and then letting $\gamma \to \infty$.

Inserting this back into (15), and writing $a^\mu = a^\mu_\parallel + a^\mu_\perp$, yields

$$\frac{1}{m\gamma^2}\left(\gamma F_\parallel t^\mu + F_\perp m^\mu\right) = a^\mu_\parallel + \frac{1}{\gamma^2}a^\mu_\perp + 2v(1-v)\left[t^\beta(\omega^\mu{}_\beta + \theta^\mu{}_\beta)\right]_\perp$$

$$+ vt^\alpha t^\beta \theta_{\alpha\beta}t^\mu + \gamma^2\frac{dv}{d\tau_0}t^\mu + v^2\frac{n'^\mu}{R'}. \tag{55}$$

Here we have then a formalism that works for an arbitrary congruence, where geodesic photons always have zero spatial curvature – by definition. If we want we can (as usual) form a single a^μ-term by multiplying the perpendicular part by γ^2. As an example, we see that for vanishing rotation and shear, the sideways force on a particle following a straight line (i.e. following a geodesic photon) is independent of the velocity.

8.2. The look-straight based curvature

In [8] a 'seeing is believing' principle is discussed. In a static spacetime (using the congruence generated by the Killing field as congruence), following a line that is *seen* as straight means that the experienced comoving sideways force will be independent of the velocity. Also a geodesic photon will follow a trajectory that *looks* straight. When there for instance is rotation of the local reference frame we may however realize that the path taken by a geodesic photon in fact will not look straight. We may however ask if it is possible to define a curvature, in more general cases than the static one (using the preferred congruence), that rests on what we *see* as straight? Indeed, as is illustrated in figure 5, we already have the necessary formalism to do this easily.
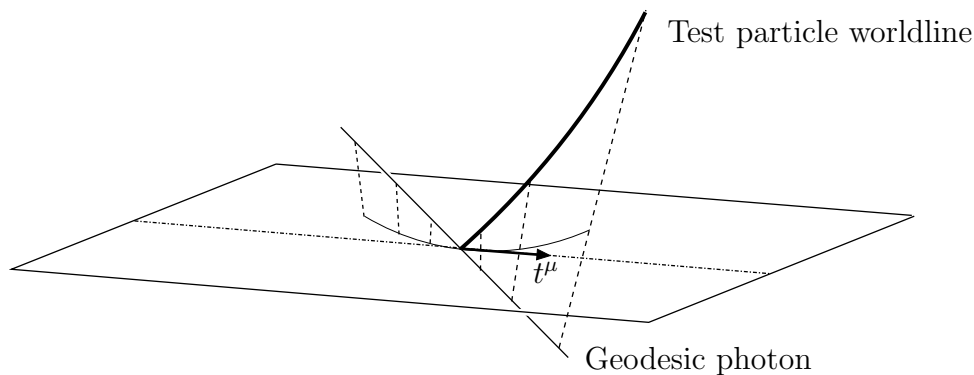

Figure 5. A 2+1 illustration, in freely falling coordinates, of a test particle following a string of congruence points (dashed worldlines), momentarily seen as aligned. The congruence points are those that are touched by an incoming (from below in time) null geodesic in the direction $-t^\mu$. In fact knowing that the upwards and downwards (in time) projection of a geodesic photon passing the slice are the same (as is obvious in coordinates adapted to the congruence), we may understand that the projected curvature of the test particle will equal the projected curvature of a geodesic photon in the $-t^\mu$-direction.

The figure illustrates that the projected curvature of a set of points that at some time was seen as aligned in a direction t^μ, in fact corresponds to the projected curvature

of a geodesic photon emitted in the direction opposite to t^μ (i.e. the $-t^\mu$ direction). From (15) we immediately find (let $t^\mu \to -t^\mu$, set $v = 1$, let $F_\parallel = F_\perp = 0$ and select the perpendicular part only)

$$0 = [a^\mu]_\perp - 2 \left[t^\beta (\omega^\mu{}_\beta + \theta^\mu{}_\beta) \right]_\perp + \frac{n^\mu}{R}. \tag{56}$$

This is then the projected curvature of those congruence lines that are momentarily *seen* as straight in the t^μ direction. We define a new curvature as

$$\frac{n''^\mu}{R''} = \frac{n^\mu}{R} + [a^\mu]_\perp - 2 \left[t^\beta (\omega^\mu{}_\beta + \theta^\mu{}_\beta) \right]_\perp. \tag{57}$$

Using this in (15), and writing $a^\mu = a^\mu_\parallel + a^\mu_\perp$, we get

$$\frac{1}{m\gamma^2} \left(\gamma F_\parallel t^\mu + F_\perp m^\mu \right) = a^\mu_\parallel + \frac{1}{\gamma^2} a^\mu_\perp + 2v(1+v) \left[t^\beta (\omega^\mu{}_\beta + \theta^\mu{}_\beta) \right]_\perp$$

$$+ v t^\alpha t^\beta \theta_{\alpha\beta} t^\mu + \gamma^2 \frac{dv}{d\tau_0} t^\mu + v^2 \frac{n''^\mu}{R''}. \tag{58}$$

So here we have the inertial force expression when we describe our motion in terms of what we *see* as straight. As always we may form a single a^μ-term if we want by dividing the parallel terms by γ^2.

We may notice that the latter definition of curvature (57) matches the definition (54) of the preceding section (curvature relative to geodesic photon), for arbitrary t^μ, if and only if

$$\left[t^\beta (\omega^\mu{}_\beta + \theta^\mu{}_\beta) \right]_\perp = 0. \tag{59}$$

This obviously holds when we have a rotationfree and shearfree congruence. Also, using an argumentation similar to that under (49), this is also necessary for (59) to hold. Notice however that (59) holds if we have (only) an isotropic expansion.

8.2.1. A comment on what looks curved The curvature as introduced in section 8.2 is good measure for the curvature as seen by a congruence observer. For the test observer that moves relative to the congruence we must also consider beaming, making small angular displacements from the forward direction shrink.

8.3. General comments

In standard optical geometry, the optical curvature radius of a spatial line that we look upon is related to the curvature radius that we experience by locally looking at the line, via a factor e^Φ. The latter two definitions (sections 8.1 and 8.2), for the particular case of a static spacetime with a Killing-adapted congruence, however both correspond exactly to the curvature that we see. The interesting thing with the standard optical curvature radius is however that it is related to a global spatial geometry. Take a trajectory, project it down onto the slice and the rescaled spatial geometry gives us the curvature. For our two latter photon-related curvatures there is in general (so far as I can see) no such global geometry (to which the curvature radius is directly related), even in the static

case. In this sense they are more abstract than the standard optical curvature radius. On the other hand the look-straight definition (in particular) is very operationally well defined regardless of there being a geometry connected to it. Lines that are seen to have a certain curvature have that very curvature, by definition. Actually, in this sense the standard optical curvature is not locally well defined operationally[21].

Looking back at the joint inertial force expression (46) for optical rescalings utilizing the projected and the new-straight curvature measures respectively, and comparing this with the latter two results, (55) and (58), we see that for a rotationfree and shearfree congruence, a geodesic photon has zero spatial curvature in all the four different formalisms. Notice however that for a rotating congruence, we cannot do the rescaling scheme (at least not without modification), since there is no well defined slicing. This excludes some of the simplest and most interesting systems where one can have use of inertial forces – such as rotating merry-go-rounds and stationary observers near a rotating object (like a Kerr black hole). The latter two definitions can however be used also for these cases.

9. Summary of the curvature measures

The perpendicular part of the inertial force equation (excepting those related to rescalings) as presented in this article is of the form

$$\frac{1}{\gamma^2}\left[\frac{Dv^\mu}{D\tau}\right]_\perp = [X_{\rm s}^\mu]_\perp + v^2\frac{n_{\rm s}^\mu}{R_{\rm s}}. \tag{60}$$

Here the index s may stand for either 'ps', 'ns', 'rp' or 'ls', corresponding to the various curvature measures as listed in order below. For these curvature measures we have $[X_{\rm s}^\mu]_\perp$ as

$$\text{Projected Straight:} \quad a_\perp^\mu + 2v\left(t^\alpha\omega^\mu{}_\alpha + [t^\alpha\theta^\mu{}_\alpha]_\perp\right) \tag{61}$$

$$\text{New-Straight:} \quad a_\perp^\mu + 2vt^\alpha\omega^\mu{}_\alpha \tag{62}$$

$$\text{Relative Photon:} \quad \frac{1}{\gamma^2}a_\perp^\mu + 2v(1-v)\left(t^\alpha\omega^\mu{}_\alpha + [t^\alpha\theta^\mu{}_\alpha]_\perp\right) \tag{63}$$

$$\text{Look Straight:} \quad \frac{1}{\gamma^2}a_\perp^\mu + 2v(1+v)\left(t^\alpha\omega^\mu{}_\alpha + [t^\alpha\theta^\mu{}_\alpha]_\perp\right). \tag{64}$$

The parallel direction of the inertial force equation is given by

$$\frac{1}{\gamma^2}\left[\frac{Dv^\mu}{D\tau}\right]_\parallel = [X_{\rm s}^\mu]_\parallel + \gamma\frac{dv}{d\tau}t^\mu. \tag{65}$$

And here

$$[X_{\rm s}^\mu]_\parallel = a_\parallel^\mu + vt^\alpha t^\beta\theta_{\alpha\beta}t^\mu. \tag{66}$$

This part is the same for all the above curvature measures.

Notice that all the four different physical ways of describing the motion relative to the reference frame yield precisely the same inertial force formalisms when using an inertial congruence (indeed there are no inertial forces then).

[21]We can never figure out what Φ is through local experiments, only its gradient can be deduced.

9.1. A comment on the different ways of defining inertial forces

As presented in this article, as is also standard for inertial forces in Newtonian mechanics, the final equation is of the form $F_{\text{real}} + F_{\text{inertial}} = ma_{\text{relative}}$. For a given physical scenario, the real force is fixed, whereas the relative acceleration, and hence the inertial forces, depend on what reference frame (congruence) we are using. Furthermore, as we have illustrated, there is more than one plausible way to define a spatial curvature for the motion of a test particle, when the reference frame is shearing[22]. This effectively means that there is more than one plausible way of describing the acceleration relative to the reference frame – hence even for a fixed reference frame there is more than one way of defining the inertial forces.

As concerns the photon related approaches, they also conform to the standard Newtonian formalism for non-shearing congruences in the limit of small velocities. We may however note that they have somewhat of a less fundamental geometrical nature to them – being more of a practical and physical nature. Consider specifically the second photon related formalism connected to what an observer comoving with the reference frame in question actually experiences visually. The apparent (inertial) forces of this formalism together with the real forces (focusing on the perpendicular part), are precisely the (apparent) forces that will make a test particle deviate from what the observer sees as straight. We understand that if we let the concept of apparent (inertial) forces be wide enough to incorporate alternative (physical) ways of measuring the *apparent* motion of a test particle – then there is room for even more definitions of inertial forces. Apart from the above mentioned alternative prescriptions, there is also a certain level of freedom concerning γ-factors, in part connected to the fact that there are two types of forces (given and received), but see also the footnote in section 3.6.

10. Three-dimensional formalism, assuming rigid congruence

We can rewrite the four-covariant inertial force formalism thus far as a purely three-dimensional formalism. For brevity let us consider a non-shearing (isotropically expanding) congruence[23], so $[t^\alpha \theta^\mu{}_\alpha]_\perp = 0$. Then the projected straight and the new-straight formalisms are identical. Since $[\theta^\mu{}_\alpha t^\alpha]_\perp = 0$ for all directions t^μ then, in freely falling coordinates locally comoving with the congruence, the spatial part of $\theta^\mu{}_\nu$ must be proportional to $\delta^i{}_j$. Also, in the coordinates in question the time components of $\theta^\mu{}_\nu$

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[22]Note incidentally that the distinction between the projected and the new type of curvature measure can be made also in non-relativistic mechanics.

[23]If we want to consider a shearing congruence in three-dimensional formalism, that is in principle no problem at all. We just define $\theta_{ij} = \frac{1}{2}(\nabla_i u_k + \nabla_k u_i)$. Here u^k is the velocity of a reference point, seen relative to a freely falling frame locally comoving with the congruence. Also ∇_i is understood to be covariant derivative with respect to the local spatial metric and lowering of indices are made using the local spatial metric. The latter can be defined, as the spatial metric on a geodesic slice (i.e. an instant in a freely falling system) orthogonal to the congruence at a single point (the point in question), without the existence of global orthogonal slices. Then we could let $\theta^\mu{}_\alpha t^\alpha \to \boldsymbol{\theta} \cdot \hat{\mathbf{t}}$ where $\boldsymbol{\theta}$ denotes the three-dimensional shear-expansion matrix.

vanishes. Defining (as is standard) $\theta = \theta^\alpha{}_\alpha$, we have thus $\theta^i{}_j \propto \frac{\theta}{3}\delta^i{}_j$. We may then write $t^\alpha t^\beta \theta_{\alpha\beta} = \frac{\theta}{3}$ (being a scalar expression this holds in general coordinates). Looking back at for instance (15), we have then

$$\frac{1}{m\gamma^2}\left(\gamma F_\| t^\mu + F_\perp m^\mu\right) = a^\mu + 2vt^\beta \omega^\mu{}_\beta + v\frac{\theta}{3}t^\mu + \gamma\frac{dv}{d\tau}t^\mu + v^2\frac{n^\mu}{R}. \qquad (67)$$

In coordinates locally comoving with the congruence[24] we have $a^\mu : (0, \mathbf{a})$, $n^\mu : (0, \mathbf{n})$ and and $vt^\mu : (0, \mathbf{v})$. To avoid confusion with the acceleration of the test particle, let us define $\mathbf{g} = -\mathbf{a}$. Also we let[25] $\omega^\mu{}_\alpha t^\alpha \to \boldsymbol{\omega} \times \hat{\mathbf{t}}$ and $\tau \to \tau_0/\gamma$ (recall that τ_0 is local time along the congruence). The three-dimensional analogue of (67) is then simply

$$\frac{1}{m\gamma^2}\left(\gamma F_\|\hat{\mathbf{t}} + F_\perp \hat{\mathbf{m}}\right) = -\mathbf{g} + 2\boldsymbol{\omega} \times \mathbf{v} + \frac{\theta}{3}\mathbf{v} + \gamma^2\frac{dv}{d\tau_0}\hat{\mathbf{t}} + v^2\frac{\hat{\mathbf{n}}}{R}. \qquad (68)$$

Notice that this formalism is defined irrespective of whether there exists any global slices orthogonal to the reference congruence. For instance we can apply it to calculate the real forces on a particle orbiting outside of the ergosphere of a Kerr black hole, using the stationary (non-rotating) observers as our reference congruence.

Multiplying the first three terms of (68) by $-m$ they can be seen as the inertial forces Acceleration, Coriolis and Expansion. The forces $F_\|$ and $F_\perp$ are the experienced (comoving) perpendicular and parallel forces respectively. If we want to consider the *given* forces $F_{c\|}$ and $F_{c\perp}$, assuming that that observers following the congruence push (or pull) the object in question, we have from Appendix C that $F_{c\|} = F_\|$ and $\gamma F_{c\perp} = F_\perp$. Indeed defining $\mathbf{F}_c = F_{c\|}\hat{\mathbf{t}} + F_{c\perp}\hat{\mathbf{m}}$, the inertial force equation becomes even simpler

$$\frac{\mathbf{F}_c}{m\gamma} = -\mathbf{g} + 2\boldsymbol{\omega} \times \mathbf{v} + \frac{\theta}{3}\mathbf{v} + \gamma^2\frac{dv}{d\tau_0}\hat{\mathbf{t}} + v^2\frac{\hat{\mathbf{n}}}{R}. \qquad (69)$$

Notice that while (68) and (69) are fully relativistically correct they are very similar to their Newtonian counterpart(s) (just set $\gamma = 1$, see also Appendix F). Notice however that τ_0 is local time in the reference frame. Considering for instance a static black hole we have $d\tau_0 = (1 - \frac{2M}{r})^{\frac{1}{2}}dt$, where t is the global (Schwarzschild time). Also space will of course in general be curved unlike in (standard) Newtonian theory.

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[24]For any specific global labeling of the congruence lines (i.e. any specific set of spatial coordinates adapted to the congruence) we can locally choose a time slice orthogonal to the congruence so that e.g. $a^\mu : (0, \mathbf{a})$. This then uniquely defines $\mathbf{a}$ at any point along the test particle trajectory.

[25]Let $\omega^\mu = \frac{1}{2}\frac{1}{\sqrt{g}}\eta_\sigma \epsilon^{\sigma\mu\gamma\rho}\omega_{\gamma\rho}$, where $g = -\text{Det}[g_{\alpha\beta}]$ and $\epsilon^{\sigma\mu\gamma\rho}$ is $+1$, -1 or 0 for $\sigma\mu\gamma\rho$ being an even, odd or no permutation of $0, 1, 2, 3$ respectively. Then we can define $\boldsymbol{\omega}$ through $\omega^\mu = (0, \boldsymbol{\omega})$ in coordinates locally orthogonal to the congruence.

Strictly speaking, what we mean by the cross product $\mathbf{a} \times \mathbf{b}$ of two three-vectors $\mathbf{a}$ and $\mathbf{b}$ is $g^{-\frac{1}{2}}\epsilon^{ijk}a_j b_k$ where the indices have been lowered with the local three-metric (assuming local coordinates orthogonal to the congruence), and g is minus the determinant of this metric. Notice that in general (for congruences with rotation) there are no global time-slices that are orthogonal to the congruence. The local three-metric corresponding to local orthogonal coordinates is however well defined everywhere anyway.

10.1. Applying the three-formalism to a rotating platform

As a simple application of the three-dimensional formalism we consider coordinates attached to a rotating platform in special relativity. Let ω_0 be the counterclockwise angular velocity of the platform, and r be the distance from the center (this distance is obviously the same whether we are corotating with the platform or not). We understand that the circumference of a circle of fixed r relative to the platform will (length contraction) be greater than the corresponding circumference, as measured on the ground, by a factor $\gamma = \gamma(\omega_0 r)$. The spatial metric in the corotating cylindrical coordinates can thus be written as

$$ds^2 = dr^2 + \frac{r^2}{1 - \omega_0^2 r^2/c^2}d\varphi^2 + dz^2. \tag{70}$$

Here c is the velocity of light. Note that this metric is well defined despite the fact that there are no time slices globally orthogonal to the reference congruence in question.

For circular motion relative to an inertial frame – the proper acceleration, as follows from (68), is given by $\gamma^2 v^2/r = \gamma^2 \omega_0^2 r$. We understand that relative to the rotating platform we have

$$\mathbf{g} = \frac{\omega_0^2 r}{1 - \omega_0^2 r^2/c^2}\hat{\mathbf{r}}. \tag{71}$$

A gyroscope orbiting with a counterclockwise angular velocity ω_0 around a circle of radius r with respect to inertial coordinates, will Thomas-precess (see e.g. [12]) with a clockwise angular velocity $\omega_{\text{gyro}} = (\gamma - 1)\omega_0$. Adding this rotation to the rotation of the reference frame and multiplying by a factor γ to take time dilation into account, it follows that with respect to an observer corotating with the platform, the gyroscope will precess with a clockwise angular velocity given by $\gamma(\omega_0 + (\gamma - 1)\omega_0) = \gamma^2 \omega_0$. We have thus the local rotation of the platform (as experienced by a locally comoving inertial observer)

$$\boldsymbol{\omega} = \frac{\omega_0}{1 - \omega_0^2 r^2/c^2}\hat{\mathbf{z}}. \tag{72}$$

Now we have the necessary tools for making calculations with respect to this reference frame.

10.2. Radial motion on the rotating platform

As a particular example we consider a wagon moving along a radially directed rail (fixed φ) on the rotating platform with constant velocity v. We are interested in what force that will act on the rail from the wagon. We note that this force is precisely minus the given force by the rail, thus $\mathbf{F}_{\text{onrail}} = -\mathbf{F}_{\text{c}}$. Letting m be the rest mass of the wagon (we assume the rotational energy of the wheels to be negligible), we have then from (69) for this simple case

$$\frac{\mathbf{F}_{\text{onrail}}}{m\gamma} = \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{v}. \tag{73}$$

Here $\gamma = \gamma(v)$. Using (71) and (72), assuming the wagon to move outwards from the center so that $\mathbf{v} = v\hat{\mathbf{r}}$, this can be written as

$$\mathbf{F}_{\text{onrail}} = m\gamma\frac{\omega_0}{1 - \omega_0^2 r^2/c^2}\left(\omega_0 r\hat{\mathbf{r}} - 2v\hat{\boldsymbol{\varphi}}\right).\tag{74}$$

Here is thus the force from the wagon on the rail. Note that the equation applies to $r < c/\omega_0$.

10.3. A few comments on the three-dimensional formalism

For typical applications where the reference congruence lines are integral curves of a timelike Killing field, we can directly use (67) and (69) respectively as equations of motion, for specified forces, to find the resulting spatial path[26]. The path can be expressed in terms of the the test particle proper time since $d\tau = d\tau_0/\gamma = ds/(v\gamma)$. For the most general case however (still assuming a non-shearing reference congruence), if we want to integrate the three-dimensional equations of motion – we need to introduce a global time parameter. In other words we need to introduce time slices in spacetime and associate with each slice a parameter t. In general $\mathbf{g}$, $\boldsymbol{\omega}$, θ and spatial distances between adjacent congruence lines will be functions of this time parameter as well as of the spatial position. Notice however that irrespective of whether the time slices are orthogonal to the congruence or not (in general they cannot be globally orthogonal to the congruence) spatial distances are always measured proper orthogonal to the congruence lines[27]. Note also that even for the stationary case, if we want to make predictions of coincidences (like whether two particles will collide or not) we need the global time parameter. For a static spacetime (like a Schwarzschild black hole), adapting the reference congruence to the static observers, there is a very simple such global time t where $dt = f(\mathbf{x})d\tau_0$ for some function $f(\mathbf{x})$. Note, however, that as local equations, (67) and (69) are directly applicable, without introducing a global time, to answer questions like for instance what perpendicular forces one gets if one follows the path of a geodesic photon.

11. A general derivation of a vector transport equation from the inertial force formalism

Jantzen et. al. have also developed a covariant inertial force formalism, see e.g [13]. They are employing various covariant differentiations of vectors defined along a spacetime trajectory. These types of covariant differentiation can readily be defined if we have a means of transporting a vector along the trajectory in question. The general idea is simple, and illustrated in figure 6.

[26]We are of course assuming that $\mathbf{g}$, $\boldsymbol{\omega}$ (for this case $\theta = 0$) and the spatial geometry are known as functions of spatial position, i.e. in terms of the labeling of the congruence lines.

[27]Thus the spatial geometry is not defined as the spatial geometry on the slice related to the global time parameter – consider for instance the example in section 10.1. Note also that the comoving coordinates used when introducing the bold face three-notation have nothing to do with the time slices related to the global time.
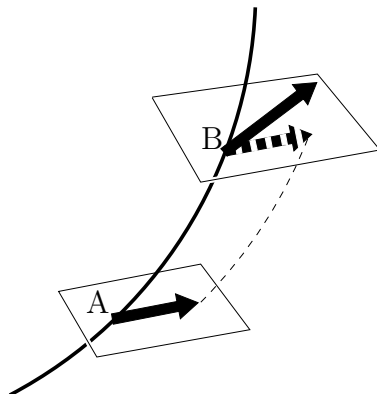
Figure 6. Vector differentiation along a timelike spacetime trajectory. The full drawn arrows correspond to the vector defined along the trajectory, for instance the momentary forward direction t^μ. The dashed arrow at B is the *transported* version of the vector at A. Forming the difference between the vectors at B and dividing by the proper time $d\tau$ along the trajectory from A to B gives us our derivative.

In particular one may define a spatial curvature and curvature direction by how fast (and in what direction) the forward direction deviates from a corresponding transported vector. The idea is that the transport law should somehow correspond to a spatial parallel transport with respect to the spatial geometry defined by the congruence. That way, the definition of spatial curvature and curvature direction is analogous to the definition in standard Riemannian three-dimensional differential geometry. In the approach of this article we started from the other end by deriving the spatial curvature measures of the various physical meanings, and we will now derive corresponding vector transports and vector differentiations.

11.1. Rigid congruence

For the case of a rigid congruence[28] the matter of spatial transport is quite intuitively reasonable. The idea is illustrated in figure 7.

It is easy to show that in the coordinates (x^k, t) of a freely falling system, locally comoving with the congruence, the velocity of the congruence points (assuming vanishing $\theta^\mu{}_\nu$) is to first order in x^k and t given by

$$v^k = \omega^k{}_j x^j + a^k t. \tag{75}$$

Knowing that the velocity of the congruence is zero to lowest order, we need not worry about length contraction and such. It is then easy to realize that the proper spacetime transport law of a vector k^μ, orthogonal to η^μ, corresponding to standard spatial parallel transport is

$$\frac{Dk^\mu}{D\tau} = \gamma\omega^\mu{}_\alpha k^\alpha + b\eta^\mu. \tag{76}$$

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[28]The congruence may rotate and accelerate but it may not shear or expand.
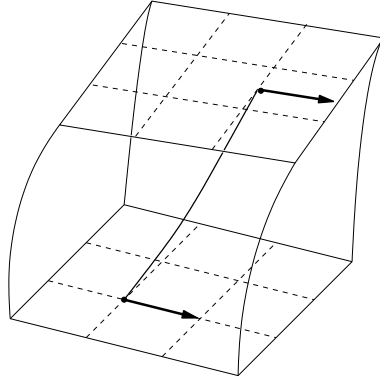
Figure 7. A 2+1 illustration of transporting a spatial vector along a worldline, seen from freely falling coordinates locally comoving with the congruence. As the coordinates attached to the grid rotate due to $\omega^\mu{}_\alpha$, so should the vector in order for it to be proper spatially transported.

Here b can easily be determined from the orthogonality of k^μ and η^{μ}[29].

11.2. General congruence

Now let us consider a congruence with non-zero expansion-shear tensor. Here there is no fixed (rigid) spatial geometry. How then to define a spacetime generalization of spatial parallel transport?

While we have no fixed spatial geometry, we still have a spatial curvature measure (of several types) given the spacetime trajectory. Suppose then that we transport a vector along a timelike worldline with vanishing spatial curvature (whichever curvature measure we choose). If the initial vector pointed in the t^μ-direction it seems natural that the parallel transported vector should keep pointing in the t^μ direction. Also, if the trajectory curves relative to a corresponding trajectory of vanishing spatial curvature, but the initial vector still pointed in the t^μ direction, the transported vector should deviate from the forward direction in the same manner as it would for a fixed geometry. We also demand of the parallel transport that the norm of the vector should be constant and it should remain orthogonal to the congruence, given that it was originally orthogonal to the congruence. Then the derivation, as concerns parallel transport of a vector momentarily parallel to t^μ, is straightforward as we illustrate in the coming two subsections.

11.2.1. The standard contravariant derivative of the forward direction Using (2), (60) and (14) we readily get

$$\left[\frac{Dt^\mu}{D\tau}\right]_\perp = \frac{\gamma}{v}[X_s^\mu]_\perp + \gamma v \frac{n_s^\mu}{R_s} - \gamma t^\alpha \omega^\mu{}_\alpha - \gamma[t^\alpha \theta^\mu{}_\alpha]_\perp - \frac{\gamma}{v}a_\perp^\mu. \tag{77}$$

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[29]We have $k^\mu \eta_\mu = 0$ which means that $\frac{Dk^\alpha}{D\tau}\eta_\alpha + k^\alpha \frac{D\eta_\alpha}{D\tau} = 0$. Contracting (76) with η_μ then readily yields $b = k^\alpha \frac{D\eta_\alpha}{D\tau}$.

So here is the perpendicular (spatial) part of how the forward direction is propagated, given the spatial curvature radius[30]. Notice that X_s^μ depends on what curvature measure we are using (see (61)-(64)).

11.2.2. The relation between spatial transport and spatial curvature   Suppose now that we have some vector $t_\parallel^\mu$ that momentarily is equal to the forward direction vector t^μ. Suppose further that we have *some* (as of yet undefined) parallel transport defined for $t_\parallel^\mu$. Then we can define a curvature measure for a trajectory, with respect to the transport in question, as

$$\gamma v \frac{n_v^\mu}{R_v} = \left[\frac{Dt^\mu}{D\tau}\right]_\perp - \left[\frac{Dt_\parallel^\mu}{D\tau}\right]_\perp. \tag{78}$$

Here the subscript 'v', stands for 'vector transport related curvature'. The definition is analogous to how one defines (may define) ordinary spatial curvature using ordinary spatial parallel transport. The γ is included since we have τ and not τ_0 on the right hand side. Using (77) and (78), making the ansatz $\frac{n_v^\mu}{R_v} = \frac{n_s^\mu}{R_s}$, hence demanding a parallel transport of the momentarily parallel vector to be such that the two types of curvature measures coincides, readily gives

$$\left[\frac{Dt_\parallel^\mu}{D\tau}\right]_\perp = \frac{\gamma}{v}[X_s^\mu]_\perp - \gamma t^\alpha \omega^\mu{}_\alpha - \gamma[t^\alpha \theta^\mu{}_\alpha]_\perp - \frac{\gamma}{v} a_\perp^\mu. \tag{79}$$

In particular for the projected and new-straight formalisms (see (61) and (62)) this yields

$$\text{Projected Straight:} \left[\frac{Dt_\parallel^\mu}{D\tau}\right]_\perp = \gamma t^\alpha \omega^\mu{}_\alpha + \gamma[t^\alpha \theta^\mu{}_\alpha]_\perp \tag{80}$$

$$\text{New-Straight:} \left[\frac{Dt_\parallel^\mu}{D\tau}\right]_\perp = \gamma t^\alpha \omega^\mu{}_\alpha - \gamma[t^\alpha \theta^\mu{}_\alpha]_\perp. \tag{81}$$

We then define the transport equation for any vector $k_\parallel^\mu$ momentarily parallel to t^μ correspondingly

$$\text{Projected Straight:} \left[\frac{Dk_\parallel^\mu}{D\tau}\right]_\perp = \gamma k_\parallel^\alpha \omega^\mu{}_\alpha + \gamma[k_\parallel^\alpha \theta^\mu{}_\alpha]_\perp \tag{82}$$

$$\text{New-Straight:} \left[\frac{Dk_\parallel^\mu}{D\tau}\right]_\perp = \gamma k_\parallel^\alpha \omega^\mu{}_\alpha - \gamma[k_\parallel^\alpha \theta^\mu{}_\alpha]_\perp. \tag{83}$$

An alternative (but equivalent) way of deriving these transport laws is to demand that the parallel transport, along a trajectory with in general non-zero spatial curvature, of a vector momentarily equaling the forward direction vector should be the same as the transport of the forward direction of a line that is *straight* with respect to the curvature measure in question.

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[30]The $\perp$-sign on the left hand side is really only necessary for the projection onto the slice, not to take away components in the t^μ-direction (since the normalization of t^μ tells us that there are no t^μ-component in $\frac{Dt^\mu}{D\tau}$).

Note that in the absence of shear, these definitions match (76). We may however note that if we instead had considered for instance the look-straight curvature, the corresponding transport would not have matched (76), even for pure rotation.

11.2.3. Spatial parallel transport of a general vector   While the just derived transport laws are sufficient for the purposes of the inertial force formalism, we may be curious to know whether we could find a transport law for *general* vectors, that corresponds to (80) and (81) for the particular case of a vector momentarily parallel to the forward direction. Indeed we can, although how we do it is quite subjective.

Let us however demand that, considering momentarily spatial vectors, the transport should be norm-preserving, and preserving orthogonality to η^μ. Also we demand that any pair of parallel transported vectors should have a fixed relative angle (in particular vectors that were initially orthogonal should remain orthogonal). In 2+1 dimensions it is obvious, concerning spatial vectors, that these considerations completely determine the parallel transport. In 3+1 dimensions there is however a freedom of (spatial) rotation around the spatial direction of motion. Here we may however take guidance from (76), and demand that in the absence of shear we should get a transport corresponding to (76). Indeed this is not generally doable as was commented upon at the end of the preceding section (11.2.2), although it will turn out to be for the case of the projected and the new-straight curvatures.

Let us assume that the parallel transport should be formulated in terms of tensors, in likeliness with (76). The tensors that we have to work with are a^μ, $\omega^\mu{}_\alpha$, $\theta^\mu{}_\alpha$, t^μ, v, η^μ, n^μ and R. From these tensors we can of course in principle form other tensors.

To insure fixed norm and angles, the transport must effectively be a spatial rotation relative to freely falling coordinates locally comoving with the congruence [31]. Given any two orthogonal spatial vectors d^μ and e^μ we can form a rotation tensor as $d^\mu e_\alpha - e^\mu d_\alpha$ [32]. For brevity we define $d^\mu \wedge e_\alpha \equiv d^\mu e_\alpha - e^\mu d_\alpha$. Several rotation tensors of this type can of course be added together to form a net rotation tensor.

There are possibly several ways to match the above criterias but the one we present below seems quite natural as concerns the new-straight and the projected curvature measures.

Looking at the different tensors available and (81) and (80) it is easy to find general transport laws, that obeys the just outlined requirements. The spatial parts of our transport laws are given below

$$\text{Projected Straight:} \left[\frac{Dk^\mu}{D\tau}\right]_\perp = \gamma k^\alpha \omega^\mu{}_\alpha + \gamma k^\alpha ([t^\beta \theta^\mu{}_\beta]_\perp \wedge t_\alpha) \tag{84}$$

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[31] The argument is similar to that in section 11.1, where length contraction will not enter. Also there will of course be a η^μ term entering to insure orthogonality.

[32] Forming $(d^\mu e_\alpha - e^\mu d_\alpha)k^\alpha$, for a spatial vector k^α, amounts to forming $\mathbf{d}(\mathbf{e}\cdot\mathbf{k}) - \mathbf{e}(\mathbf{d}\cdot\mathbf{k})$ in (spatial) bold-face notation. This is a so called vector triple product and equals $(\mathbf{e}\times\mathbf{d})\times\mathbf{k}$. Thus $d^\mu e_\alpha - e^\mu d_\alpha$ is a rotation tensor.

$$\text{New-Straight:} \qquad \left[\frac{Dk^\mu}{D\tau}\right]_\perp = \gamma k^\alpha \omega^\mu{}_\alpha - \gamma k^\alpha([t^\beta \theta^\mu{}_\beta]_\perp \wedge t_\alpha). \tag{85}$$

Defining the transport law in such a way that a vector originally orthogonal to the congruence remains orthogonal to the congruence, we can add a term $\eta^\mu k^\alpha \frac{D\eta_\alpha}{D\tau}$ (analogous to what we did in section 11.1) to the right hand side of (84) and (85). That way we may remove the $\perp$ sign on the left hand side (which was anyway there only for projection, not for orthogonality to t^μ), and express the full transport equations.

Note that rather than $k^\alpha \omega^\mu{}_\alpha$ we might for instance have tried $k^\alpha(t^\beta \omega^\mu{}_\beta \wedge t_\alpha)$. These would both give the right transport equation when $k^\mu = t^\mu$ momentarily, while in general being different for other vectors k^μ. The latter rotation version would introduce no rotation at all around the direction of motion (the rotation vector is given by $\boldsymbol{\omega} \times \boldsymbol{t}$, where $\boldsymbol{\omega}$ is the rotation three vector corresponding to the rotation tensor $\omega^\mu{}_\alpha$), as seen from an inertial system. This is however not really what we *want*. For a static rotating grid it seems obvious that the parallel transport should coincide with standard spatial parallel transport. Hence if the congruence rotates around the direction of motion (seen from an inertial system) so should a parallel transported vector. We should thus use $k^\alpha \omega^\mu{}_\alpha$ rather than $k^\alpha(t^\beta \omega^\mu{}_\beta \wedge t_\alpha)$. As regards the $\theta^\mu{}_\alpha$-term, what we want is not as clear. The way that we have chosen gives the minimal rotation needed (seen from an inertial system) to get the transport right.

Note that the ambiguity in rotation around the spatial direction of motion, for parallel transport of a general vector, has no impact on the discussion of inertial forces. Here we are always concerned with rotation of vectors momentarily in the forward direction, for which case there is no ambiguity. The general transport laws can however be used in other contexts. In particular one may use them when developing a relativistic three-dimensional formalism of gyroscope precession relative to a given reference frame. In such a formalism, see [12], the occurrence of for instance terms of the type $\gamma k^\alpha([t^\beta \theta^\mu{}_\beta]_\perp \wedge t_\alpha)$ follows naturally, independent of what spatial parallel transport we consider. Thus the form of (84) and (85) fits well with the formalism of three-dimensional relativistic gyroscope precession.

11.2.4. Covariant differentiation along trajectory Having derived the transport laws, the corresponding covariant differentiations along a trajectory follows immediately. Including the η^μ-component as discussed under (85) we simply get

$$\frac{D_{\text{ps}}k^\mu}{D_{\text{ps}}\tau} = \frac{Dk^\mu}{D\tau} - \gamma k^\alpha \omega^\mu{}_\alpha - \gamma k^\alpha([t^\beta \theta^\mu{}_\beta]_\perp \wedge t_\alpha) - \eta^\mu k^\alpha \frac{D\eta_\alpha}{D\tau} \tag{86}$$

$$\frac{D_{\text{ns}}k^\mu}{D_{\text{ns}}\tau} = \frac{Dk^\mu}{D\tau} - \gamma k^\alpha \omega^\mu{}_\alpha + \gamma k^\alpha([t^\beta \theta^\mu{}_\beta]_\perp \wedge t_\alpha) - \eta^\mu k^\alpha \frac{D\eta_\alpha}{D\tau}. \tag{87}$$

Here $\frac{D\eta_\alpha}{D\tau}$ is readily given by (9) and (14). Notice however that for the purposes of the inertial force formalism presented here, only the projected part of these equations is of importance and only when applied to a vector momentarily parallel to the forward direction.

12. Reformulating the inertial force formalism

Consider a rigid Cartesian reference system that rotates and possibly accelerates in Newtonian mechanics. The law of motion can then be expressed relative to the reference system as (see also Appendix F)

$$\frac{\boldsymbol{F}}{m} = -\frac{1}{m}\boldsymbol{F}_{\text{inertial}} + \frac{dv}{dt}\boldsymbol{t} + v^2\frac{\boldsymbol{n}}{R}. \tag{88}$$

Here v and R are the velocity and spatial curvature relative to the reference system, analogous to the approach of the preceding section. Alternatively we could express (88) as

$$\frac{\boldsymbol{F}}{m} = -\frac{1}{m}\boldsymbol{F}_{\text{inertial}} + \frac{1}{m}\frac{d\boldsymbol{p}}{dt}. \tag{89}$$

Here $\boldsymbol{p} \equiv m\boldsymbol{v}$ is the three-momentum relative to the reference system.

The question arises if we could do something similar in the general relativistic scheme? Indeed we already have the necessary tools to transport relativistic three-momentum, and do a differentiation corresponding to $\frac{d\mathbf{p}}{dt}$. When only concerned with inertial forces, there is however a more direct way (allowing some overlap with the preceding section) as will be presented below.

12.1. The reformulation, with the corresponding transport in implicit form

Let us introduce the relativistic three-momentum relative to the congruence as $\bar{p}^{\mu} \equiv P^{\mu}{}_{\alpha}p^{\alpha}$ (the bar here has nothing to do with the bar indicating new-straight curvature and curvature direction). For the particular case of special relativity, for an inertial congruence, (12) then gives us

$$\frac{1}{m\gamma^2}\frac{D\bar{p}^{\mu}}{D\tau} = \gamma\frac{dv}{d\tau}t^{\mu} + v^2\frac{n^{\mu}}{R}. \tag{90}$$

By analogy with this relation we now *define* a covariant differentiation of three-momentum along a curve as[33]

$$\frac{1}{m\gamma^2}\frac{D_{\text{s}}\bar{p}^{\mu}}{D_{\text{s}}\tau} = \gamma\frac{dv}{d\tau}t^{\mu} + v^2\frac{n_{\text{s}}^{\mu}}{R_{\text{s}}}. \tag{91}$$

For a general inertial force equation of the form

$$\frac{1}{m\gamma^2}\left(\gamma F_{\|}t^{\mu} + F_{\perp}m^{\mu}\right) = X_{\text{s}}^{\mu} + \gamma\frac{dv}{d\tau}t^{\mu} + v^2\frac{n_{\text{s}}^{\mu}}{R_{\text{s}}}. \tag{92}$$

we can thus write alternatively

$$\frac{1}{m\gamma^2}\left(\gamma F_{\|}t^{\mu} + F_{\perp}m^{\mu}\right) = X_{\text{s}}^{\mu} + \frac{1}{m\gamma^2}\frac{D_{\text{s}}\bar{p}^{\mu}}{D_{\text{s}}\tau}. \tag{93}$$

Here we have then a reformulation of the inertial force formalism, although the transport equation connected to the derivative is left implicit. We can however derive it from the

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[33]Considering that $\frac{D\bar{p}^{\mu}}{D\tau}$ has an η^{μ} component, one might think that also $\frac{D_{\text{s}}\bar{p}^{\mu}}{D_{\text{s}}\tau}$ should have it. The latter is however intended to be a differentiation between two infinitesimally different vectors that are exactly orthogonal to η^{μ} after some infinitesimal time. It is then easy to show that it should not contain any explicit η^{μ} component.

above formalism, analogous to the derivation of the preceding section. We do this in the following section.

12.2. Re-deriving the transport equation

We have by definition

$$\frac{D_s \bar{p}^\mu}{D_s \tau} \equiv \frac{D\bar{p}^\mu}{D\tau} - \frac{D\bar{p}^\mu_\parallel}{D\tau}. \tag{94}$$

Here $\bar{p}^\mu_\parallel$ is understood to be a vector that is momentarily parallel to $\bar{p}^\mu$ and then 'parallel' transported with respect to the congruence (and the curvature measure in question). This we can now use to derive the transport equation. First we write (93) as

$$\frac{1}{m\gamma^2} P^\mu{}_\alpha \frac{Dp^\alpha}{D\tau} = X_s^\mu + \frac{1}{m\gamma^2} \frac{D_s \bar{p}^\mu}{D_s \tau}. \tag{95}$$

Using the definitions of $\bar{p}^\mu$ and $P^\mu{}_\nu$ together with (94) it is then easy to show that

$$\frac{1}{m\gamma^2} \frac{Dp^\mu_\parallel}{D\tau} = X_s^\mu - a^\mu - vt^\alpha \nabla_\alpha \eta^\mu + \eta^\mu t^\alpha (\eta^\rho + vt^\rho) \nabla_\rho \eta_\alpha. \tag{96}$$

Using (61), (62) and (66) together with (9) and (14) we readily find the projected version of this equation for the projected and new-straight formalisms respectively

$$\text{Projected:} \qquad \frac{1}{m\gamma^2} P^\mu{}_\alpha \frac{D\bar{p}^\alpha_\parallel}{D\tau} = v\omega^\mu{}_\alpha t^\alpha + v \left[\theta^\mu{}_\alpha t^\alpha \right]_\perp \tag{97}$$

$$\text{New-straight:} \qquad \frac{1}{m\gamma^2} P^\mu{}_\alpha \frac{D\bar{p}^\alpha_\parallel}{D\tau} = v\omega^\mu{}_\alpha t^\alpha - v \left[\theta^\mu{}_\alpha t^\alpha \right]_\perp. \tag{98}$$

These are a perfect match with (82) and (83) (substituting $k^\mu \to \bar{p}^\alpha$). Note that for this particular type of transport there are no ambiguities, since the vector we are transporting is momentarily parallel to the direction of motion.

13. The Jantzen et. al. approach revisited

Jantzen et. al. (see e.g. [14]), are using four different definitions of covariant differentiation along curve. In the language of this article, assuming the vector in question to be momentarily orthogonal to η^μ[34], the definitions are[35]

$$\frac{D_{\text{fw}} k^\mu}{D\tau} = P^\mu{}_\beta \frac{Dk^\beta}{D\tau} \tag{99}$$

$$\frac{D_{\text{cfw}} k^\mu}{D\tau} = P^\mu{}_\beta \frac{Dk^\beta}{D\tau} - \gamma \omega^\mu{}_\alpha k^\alpha \tag{100}$$

$$\frac{D_{\text{lie}} k^\mu}{D\tau} = P^\mu{}_\beta \frac{Dk^\beta}{D\tau} - \gamma \left(\omega^\mu{}_\alpha + \theta^\mu{}_\alpha \right) k^\alpha \tag{101}$$

$$\frac{D_{\text{lie}\flat} k^\mu}{D\tau} = P^\mu{}_\beta \frac{Dk^\beta}{D\tau} - \gamma \left(\omega^\mu{}_\alpha - \theta^\mu{}_\alpha \right) k^\alpha. \tag{102}$$

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[34]If the vector has a time component we should add a term $\gamma k^\alpha \eta_\alpha a^\mu$ on the right hand side of (101).

[35]Note in particular that they are using a different convention regarding the sign of $\omega^\mu{}_\alpha$, here we are however using the convention of this article.

The subscripts are short for 'Fermi-Walker', 'Co-rotating Fermi-Walker', 'Lie' and 'covariant Lie'. Note that while 'fw' really stands for Fermi-Walker (99) is not the standard Fermi-Walker derivative.

Defining $\bar{p}^\mu \equiv m P^\mu{}_\alpha v^\alpha$, we have $v^\mu = \gamma \eta^\mu + \frac{1}{m}\bar{p}^\mu$ and thus

$$\frac{1}{\gamma^2}P^\mu{}_\alpha \frac{Dv^\alpha}{D\tau} = \frac{1}{\gamma^2}P^\mu{}_\alpha \frac{D}{D\tau}(\gamma \eta^\alpha) + \frac{1}{m\gamma^2}P^\mu{}_\alpha \frac{D\bar{p}^\alpha}{D\tau}. \tag{103}$$

We have also

$$\frac{D\eta^\mu}{D\tau} = v^\alpha \nabla_\alpha \eta^\mu \tag{104}$$

$$= \gamma(\eta^\alpha + vt^\alpha)(\theta^\mu{}_\alpha + \omega^\mu{}_\alpha - a^\mu \eta_\alpha) \tag{105}$$

$$= \gamma a^\mu + \gamma v(\theta^\mu{}_\alpha + \omega^\mu{}_\alpha). \tag{106}$$

This we may use in (103) together with (99)-(102) (subsequently), substituting $k^\rho \to \bar{p}^\rho$. Letting 'tem' denote 'fw','cfw','lie' or 'lie♭', we immediately retrieve the result of Jantzen et. al.

$$\frac{1}{\gamma^2}P^\mu{}_\alpha \frac{Dv^\alpha}{D\tau} = \frac{1}{m\gamma^2}\frac{D_\text{tem}\bar{p}^\mu}{D\tau} + X^\mu_\text{tem}. \tag{107}$$

Here

$$X^\mu_\text{tem} = a^\mu - vH_\text{tem}{}^\mu{}_\alpha t^\alpha. \tag{108}$$

Here in turn, $H_\text{tem}{}^\mu{}_\alpha$ is given by

$$H_\text{fw}{}^\mu{}_\alpha = -\omega^\mu{}_\alpha - \theta^\mu{}_\alpha \tag{109}$$

$$H_\text{cfw}{}^\mu{}_\alpha = -2\omega^\mu{}_\alpha - \theta^\mu{}_\alpha \tag{110}$$

$$H_\text{lie}{}^\mu{}_\alpha = -2\omega^\mu{}_\alpha - 2\theta^\mu{}_\alpha \tag{111}$$

$$H_\text{lie♭}{}^\mu{}_\alpha = -2\omega^\mu{}_\alpha. \tag{112}$$

Already here we have the inertial force formalism. In the coming subsection we will compare the two formalisms.

Jantzen et. al. has also considered an inertial force formalism in terms of curvatures as experienced by the comoving observer [15]. The idea is essentially to study how fast, and in what direction, the incoming congruence points are changing their velocity relative to a comoving reference frame of gyroscopes.

13.1. Comparing the formalisms

We can write (95) as

$$\frac{1}{m\gamma^2}\frac{Dv^\mu}{D\tau} = \frac{1}{m\gamma^2}\frac{D_\text{s}\bar{p}^\mu}{D_\text{s}\tau} + X^\mu_\text{s}. \tag{113}$$

Looking at (61), (62) (we skip the relative photon and look-straight curvature measures) and (66) we have X^μ_s as

$$\text{Projected Straight:} \quad X^\mu_\text{ps} = a^\mu + 2v\left(t^\alpha \omega^\mu{}_\alpha + [t^\alpha \theta^\mu{}_\alpha]_\perp\right) + vt^\alpha t^\beta \theta_{\alpha\beta}t^\mu \tag{114}$$

$$\text{New-Straight:} \quad X^\mu_\text{ns} = a^\mu + 2vt^\alpha \omega^\mu{}_\alpha + vt^\alpha t^\beta \theta_{\alpha\beta}t^\mu \tag{115}$$

We may compare these two equations with (111) and (112). We see that as regards the perpendicular part, the new-straight formalism of this article corresponds to the lie♭-formalism and the projected straight formalism corresponds to the lie-formalism[36]. The corresponding parallel parts are however not equal.

How one deals with the parallel part is to a large degree a matter of taste. In this article we have defined parallel transport in such a way that the norm of the parallel transported vector is preserved. This is a natural definition if we want to connect directly to changes in the local speed v. Consider for instance an isotropically expanding universe with a particle moving along a straight line (here all the curvature measures coincide) with constant local speed (this requires a forward thrust) relative to the preferred congruence. With parallel transport as defined in this article we have then $\frac{D\bar{p}^{\mu}}{D\tau} = 0$. In this view the forward thrust cancels the fictitious expansion force.

The philosophy regarding the perpendicular part of the transport equations are also a little different. We have here considered transport equations that by definition are not altering the angles between transported vectors, which is not generally the case in the approach of Jantzen et. al. Again this is a matter of taste, and it has no impact at all on the discussion of inertial forces since we are anyway only interested in the transport of a vector locally aligned with the forward direction.

The biggest difference in our approaches is that we have here started from various physically defined curvature measures, and derived an inertial force formalism from this. Only after this was done have we considered the notion of spatial parallel transport with respect to the congruence. Jantzen et. al. on the other hand start from various transport equations and derive the formalism and curvature measures from this.

Considering the new-straight formalism the Jantzen et. al. approach is not really applicable. While the curvature connected to the *lie♭*-transport in fact corresponds to the new-straight curvature, the connection appears coincidental. The physical meaning of this curvature (related to minimizing the local integrated distance) has not previously been discussed (to the author's knowledge). Neither has any formalism previously been presented (again to the author's knowledge) employing this curvature measure explicitly.

14. Summary and conclusion

The inertial force formalism as developed here was initially inspired by the works of Abramowicz et. al. who have employed a rescaled version of space(time) to study inertial forces. We have here extended the formalism of inertial forces in rescaled spacetimes to include arbitrary hypersurface-forming congruences (applicable to any spacetime). A generalization has earlier been studied in [6] using a different philosophy, but see [13] for criticism. We find that the inertial force formalism is very similar in the rescaled and the standard spacetime and that the difference lies mainly in how the γ-factors enter. The main part of this article has turned out to be more connected to the work of Jantzen et. al. A novelty with the approach of this paper is that we are starting from

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[36]The latter is expected considering the way the Lie derivative entered the derivation of section 3.

various physically defined spatial curvature measures, and are using these to describe the local motion of a test particle, and derive a corresponding inertial force formalism. In particular we introduce a new curvature measure that we denote new-straight curvature. This measure is defined in such a way that, even when we have a shearing congruence, following a straight line with respect to the new curvature measure means taking the shortest path relative to the spatial geometry defined by the congruence (which is actually not the case for the standard projected curvature). This provides us with a natural way of extending the optical geometry, to include the most general hypersurface-forming reference frames, while keeping the most basic features. Indeed we show that as regards photons, the new-straight curvature is strongly connected to Fermat's principle. These considerations and others will be further commented upon in a companion paper [3] on generalizing the optical geometry.

We have also considered a pair of more unorthodox curvature measures, the curvature relative to that of a geodesic photon and the look-straight curvature. Likely these will have even less practical import than the projected and the new-straight curvature measures, but they serve as examples of the variety of different curvature measures, and corresponding inertial force formalisms, that one may introduce. They also illustrate how one may apply the inertial force formalism to answer some particular questions in physics.

From the derived curvature measures, we have derived spacetime transport laws for vectors, along a test particle worldline, corresponding to spatial parallel transport with respect to the congruence. These transport laws can for example be used to derive an expression for how a gyroscope precesses relative to the reference congruence.

We have not in this paper spent much time on explaining for instance *why* the sideways force increases by a γ^2-factor if we follow a straight line in a static spacetime. For such considerations we refer to a companion paper [16]. There we rely on simple principles such as time dilation and the equivalence principle and derive the relativistic three-dimensional form of the inertial force equation (68) using no four-covariant formalism at all. While this paper is considerably more formal in its approach, we have tried to employ an (in the author's mind) more accessible mathematical notation than that employed by Jantzen et. al.

The explicit three-formalism as presented for shearfree (but isotropically expanding, accelerating and rotating) reference frames is, to the author's knowledge, also novel.

Appendix A. The kinematical invariants of the congruence

The kinematical invariants of a congruence of worldlines of four-velocity η^μ are defined as (see e.g. [5])

$$a_\mu \quad = \eta^\alpha \nabla_\alpha \eta_\mu \tag{A.1}$$

$$\theta \quad = \nabla_\alpha \eta^\alpha \tag{A.2}$$

$$\sigma_{\mu\nu} = \frac{1}{2} \left(\nabla_\rho \eta_\mu P^\rho{}_\nu + \nabla_\rho \eta_\nu P^\rho{}_\mu \right) - \frac{1}{3} \theta P_{\mu\nu} \tag{A.3}$$

$$\omega_{\mu\nu} = \frac{1}{2} \left(\nabla_\rho \eta_\mu P^\rho{}_\nu - \nabla_\rho \eta_\nu P^\rho{}_\mu \right). \tag{A.4}$$

In order of appearance these objects denote the acceleration vector, the expansion scalar, the shear tensor and the rotation tensor. We will also employ what we may denote the expansion-shear tensor

$$\theta_{\mu\nu} = \frac{1}{2} \left(\nabla_\rho \eta_\mu P^\rho{}_\nu + \nabla_\rho \eta_\nu P^\rho{}_\mu \right). \tag{A.5}$$

Appendix B. Rewriting f^μ in terms of experienced (comoving) forward and sideways forces

Consider a freely falling frame, locally comoving with η^μ, with a particle moving relative to this frame. In the coordinates of the inertial frame, the particle is acted upon by a force f^μ. This force may be decomposed as

$$f^\mu = f^0 \eta^\mu + f_\| t^\mu + f_\perp m^\mu. \tag{B.1}$$

Here m^μ is a normalized spatial vector orthogonal to t^μ.

The corresponding four-force in a system locally comoving with the particle, with velocity vt^μ, is related to f^μ simply via the Lorentz transformation. We may then align the first spatial coordinate with the direction of motion, and the second with the direction of the perpendicular force (m^μ). Denoting the components of the corresponding decomposition in the comoving system by (capital) F, using the fact that $F^0 = 0$, the Lorentz-transformation gives us

$$0 \quad = \gamma(f^0 - v f_\|) \tag{B.2}$$
$$F_\| = \gamma(f_\| - v f^0) \tag{B.3}$$
$$F_\perp = f_\perp. \tag{B.4}$$

From the first and second equation above follows that $f_\| = \gamma F_\|$. Using (B.1), we have then

$$P^\mu{}_\alpha f^\alpha = \gamma F_\| t^\mu + F_\perp m^\mu. \tag{B.5}$$

Here $F_\|$ is the experienced forward thrust (by a comoving observer), and $F_\perp$ is the experienced sideways thrust. Note that while we proved the equality in a certain system, both sides are tensorial and thus it holds in any coordinate system.

Appendix C. Expressing the four-acceleration in terms of the forces given by the congruence observers

Letting $p^\mu = m v^\mu$ denote the four-momentum of a particle we have

$$\frac{Dv^\mu}{D\tau} = \frac{1}{m} \frac{Dp^\mu}{D\tau} \tag{C.1}$$
$$= \frac{\gamma}{m} \frac{Dp^\mu}{D\tau_0}. \tag{C.2}$$

Here τ_0 is local time along the congruence. Looking at the right hand side in the coordinates of an inertial system locally comoving with the congruence, we see that the spatial part expresses momentum transfer per unit time i.e. force. So denoting the given forces parallel and perpendicular to the direction of motion by $F_{c\|}$ and $F_{c\perp}$ we have by definition

$$P^\mu{}_\alpha \frac{Dp^\alpha}{D\tau_0} \equiv F_{c\|}t^\mu + F_{c\perp}m^\mu. \tag{C.3}$$

Hence we have

$$\frac{1}{\gamma^2} P^\mu{}_\alpha \frac{Dv^\alpha}{D\tau} = \frac{1}{\gamma m}(F_{c\|}t^\mu + F_{c\perp}m^\mu). \tag{C.4}$$

We may note by comparison with (B.5) that $F_{c\|} = F_\|$ and $F_{c\perp} = F_\perp/\gamma$.

Appendix D. Conformal transformations, covariant differentiation and the rescaled kinematical invariants

Consider a conformal transformation $\tilde{g}_{\mu\nu} = e^{-2\Phi}g_{\mu\nu}$. Let k^μ be a general vector field and $\tilde{k}^\mu = e^\phi k^\mu$ its rescaled analogue. We have then

$$\tilde{\nabla}_\mu \tilde{k}^\nu = \partial_\mu(e^\Phi k^\nu) + \tilde{\Gamma}^\nu{}_{\mu\alpha} e^\Phi k^\alpha. \tag{D.1}$$

Evaluated in a system in free fall relative to the original spacetime (so $\partial_\mu \to \nabla_\mu$), we have

$$\tilde{\Gamma}^\mu{}_{\alpha\beta} \overset{*}{=} \frac{1}{2}\tilde{g}^{\mu\rho}\left(\nabla_\alpha \tilde{g}_{\rho\beta} + \nabla_\beta \tilde{g}_{\rho\alpha} - \nabla_\rho \tilde{g}_{\alpha\beta}\right) \tag{D.2}$$

$$\overset{*}{=} \dots$$

$$\overset{*}{=} -\left[g^\mu{}_\alpha(\nabla_\beta\Phi) + g^\mu{}_\beta(\nabla_\alpha\Phi) - g^{\mu\rho}(\nabla_\rho\Phi)g_{\alpha\beta}\right]. \tag{D.3}$$

Using this in (D.1), evaluated in a freely falling system relative to the original spacetime, we readily find

$$\tilde{\nabla}_\mu \tilde{k}^\nu = e^\Phi\left(\nabla_\mu k^\nu - g^\nu{}_\mu k^\alpha \nabla_\alpha\Phi + k_\mu g^{\nu\rho}\nabla_\rho\Phi\right). \tag{D.4}$$

This holds in originally freely falling coordinates. Since both sides are tensorial it holds in any coordinates. A corresponding expression for a covariant vector $\tilde{k}_\mu = e^{-\Phi}k_\mu$ is given by

$$\tilde{\nabla}_\mu \tilde{k}_\nu = e^{-\Phi}\left(\nabla_\mu k_\nu - g_{\mu\nu}k^\alpha \nabla_\alpha\Phi + k_\mu \nabla_\nu\Phi\right). \tag{D.5}$$

Now let us apply this to the congruence invariants. The invariants are defined according to (A.1)-(A.5). Using (D.5) and (D.4), assuming a $(-,+,+,+)$ metric, we readily find the corresponding rescaled analogues

$$\tilde{a}^\mu = e^{2\Phi}(a^\mu - P^{\mu\alpha}\nabla_\alpha\Phi) \tag{D.6}$$

$$\tilde{\theta} = e^\Phi(\theta - 3\eta^\alpha\nabla_\alpha\Phi) \tag{D.7}$$

$$\tilde{\sigma}_{\mu\nu} = e^{-\Phi}\sigma_{\mu\nu} \tag{D.8}$$

$$\tilde{\omega}_{\mu\nu} = e^{-\Phi}\omega_{\mu\nu} \tag{D.9}$$

$$\tilde{\theta}_{\mu\nu} = e^{-\Phi}\left(\theta_{\mu\nu} - P_{\mu\nu}\eta^\alpha\nabla_\alpha\Phi\right). \tag{D.10}$$

It may also be convenient to know how the covariant derivative of a vector defined along a curve transforms. Suppose then that we have a vector k^μ defined along a trajectory of four-velocity v^μ. Let $\tilde{k}^\mu = e^\Phi k^\mu$ and $\tilde{v}^\mu = e^\Phi v^\mu$. Considering an arbitrary smooth extension of the vector k^μ around the worldline, we can write[37] $\frac{\tilde{D}\tilde{k}^\mu}{\tilde{D}\tilde{\tau}} = \tilde{v}^\alpha \tilde{\nabla}_\alpha \tilde{k}^\mu$, and apply (D.4)

$$\frac{\tilde{D}\tilde{k}^\mu}{\tilde{D}\tilde{\tau}} = \tilde{v}^\alpha \tilde{\nabla}_\alpha \tilde{k}^\mu \tag{D.11}$$

$$= \dots$$

$$= e^{2\Phi}\left(\frac{Dk^\mu}{D\tau} - (v^\mu k^\rho - v^\alpha k_\alpha g^{\mu\rho})\nabla_\rho\Phi\right). \tag{D.12}$$

In particular, considering $k^\mu = v^\mu$, we get the transformation of the four-acceleration

$$\frac{\tilde{D}\tilde{v}^\mu}{\tilde{D}\tilde{\tau}} = e^{2\Phi}\left(\frac{Dv^\mu}{D\tau} - (g^{\mu\rho} + v^\mu v^\rho)\nabla_\rho\Phi\right). \tag{D.13}$$

Equivalently we may write (D.13) as

$$\frac{\tilde{D}^2 x^\mu}{\tilde{D}\tilde{\tau}^2} = e^{2\Phi}\frac{D^2 x^\mu}{D\tau^2} - \frac{dx^\mu}{d\tilde{\tau}}\frac{dx^\rho}{d\tilde{\tau}}\tilde{\nabla}_\rho\Phi - \tilde{g}^{\mu\rho}\tilde{\nabla}_\rho\Phi \tag{D.14}$$

So here is how the four-acceleration with respect to the rescaled spacetime is related to the four-acceleration with respect to the standard spacetime.

Appendix E. The acceleration of the generating observers in optical geometry

We have

$$\eta_\mu = -e^\Phi \nabla_\mu t \tag{E.1}$$

$$\eta^\alpha \eta_\alpha = -1. \tag{E.2}$$

From the normalization follows that $\eta^\alpha \nabla_\mu \eta_\alpha = 0$.

$$0 = \eta^\alpha \nabla_\mu \eta_\alpha \tag{E.3}$$

$$= -\eta^\alpha \nabla_\mu(e^\Phi \nabla_\alpha t) \tag{E.4}$$

$$= -\eta^\alpha(e^\Phi \nabla_\mu\Phi\nabla_\alpha t + e^\Phi\nabla_\mu\nabla_\alpha t) \tag{E.5}$$

$$= -\nabla_\mu\Phi - e^\Phi\eta^\alpha\nabla_\mu\nabla_\alpha t. \tag{E.6}$$

This will useful when we evaluate the four-acceleration below

$$\frac{D\eta_\mu}{D\tau} = \eta^\alpha \nabla_\alpha \eta_\mu \tag{E.7}$$

$$= -\eta^\alpha \nabla_\alpha(e^\Phi \nabla_\mu t) \tag{E.8}$$

$$= -\eta^\alpha(e^\Phi \nabla_\alpha\Phi\nabla_\mu t + e^\Phi\nabla_\alpha\nabla_\mu t) \tag{E.9}$$

$$= \eta_\mu\eta^\alpha\nabla_\alpha\Phi + \nabla_\mu\Phi \tag{E.10}$$

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[37] We could just do an analogous derivation to that leading to (D.4) but for $\frac{Dk^\mu}{D\tau}$. Using the trick of extending the vector around the trajectory we can however use the already derived formalism and save a little time.

$$= (\eta^{\alpha}\eta_{\mu} + \delta^{\alpha}{}_{\mu})\nabla_{\alpha}\Phi \tag{E.11}$$

$$= P^{\alpha}{}_{\mu}\nabla_{\alpha}\Phi. \tag{E.12}$$

We notice that the right hand side is orthogonal to η^{μ}, as it must. While the above derivation by itself had nothing to do with rescalings of spacetime, we can still in principle consider an optically rescaled spacetime $\tilde{g}_{\mu\nu} = e^{-2\Phi}g_{\mu\nu}$, where $\tilde{\eta}_{\mu} = -\tilde{\nabla}_{\mu}t$. Then just setting tildes on everything above (for the case $\Phi = 0$) it immediately follows that $\frac{\tilde{D}\tilde{\eta}^{\mu}}{\tilde{D}\tilde{\tau}} = 0$. This is very intuitively reasonable, because in the rescaled spacetime there is no time dilation, thus being at rest must maximize the proper time. Therefore, in the rescaled spacetime we have $\frac{\tilde{D}\tilde{\eta}^{\mu}}{\tilde{D}\tilde{\tau}} = 0$. If we use this, then (E.12) follows from (D.6).

Appendix F. A note on the Newtonian analogue

In typical inertial force applications in the Newtonian theory, one assumes a rigid reference frame that has an acceleration $\mathbf{A}_0$ of the origin and a rotation $\boldsymbol{\omega}$ that may change over time (non-zero $\dot{\boldsymbol{\omega}}$). Following e.g. [17] we have

$$\mathbf{F} - m\mathbf{A}_0 \underbrace{-2m\boldsymbol{\omega} \times \mathbf{v}'}_{\mathbf{F}_{\text{Cor}}} \underbrace{-m\dot{\boldsymbol{\omega}} \times \mathbf{r}'}_{\mathbf{F}_{\text{trans}}} \underbrace{-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')}_{\mathbf{F}_{\text{centrif}}} = m\mathbf{a}' \tag{F.1}$$

Here $\mathbf{F}$ is the real force and a prime means that the quantity is connected to the reference frame in question (which is not inertial in general). In particular $\mathbf{a}'$ is the acceleration relative to the reference frame. From now on we will however let $\mathbf{a}' = \mathbf{a}_{\text{rel}}$ and drop the primes to conform with the notation of this article. In the above expression $\mathbf{F}_{\text{Cor}}$ is the Coriolis force, $\mathbf{F}_{\text{centrif}}$ is the centrifugal force and $\mathbf{F}_{\text{transv}}$ is the transverse force. While the former two forces have standard names, the latter does not appear to have a universally accepted name (as pointed out in [18]) – in fact in [18] it is denoted the Euler force.

Considering motion along a special path of local curvature direction $\hat{\mathbf{n}}$ and curvature radius R with respect to the reference frame, we can alternatively express the relative acceleration as

$$\mathbf{a}_{\text{rel}} = \frac{dv}{dt}\hat{\mathbf{t}} + v^2\frac{\hat{\mathbf{n}}}{R} \tag{F.2}$$

Here $\hat{\mathbf{t}}$ is the (normalized) direction of motion with respect to the reference frame. Using (F.2) the inertial force equation then takes the form

$$\mathbf{F} - m\mathbf{A}_0 + \mathbf{F}_{\text{Cor}} + \mathbf{F}_{\text{transv}} + \mathbf{F}_{\text{centrif}} = \frac{dv}{dt}\hat{\mathbf{t}} + v^2\frac{\hat{\mathbf{n}}}{R} \tag{F.3}$$

The proper (relative to an inertial frame) acceleration of a certain point $\mathbf{r}'$ fixed relative to the reference frame can be found from (F.3). Note that the Coriolis force and the relative acceleration (the right hand side of (F.3)) are zero for this case, thus we have

$$m\mathbf{a}_{\text{reference}} = m\mathbf{A}_0 - \mathbf{F}_{\text{transv}} - \mathbf{F}_{\text{centrif}} \tag{F.4}$$

Using this in (F.3) and moving terms around, also using the explicit form of the Coriolis force, we readily find

$$\frac{1}{m}\mathbf{F} = \mathbf{a}_{\text{reference}} + 2\boldsymbol{\omega} \times \mathbf{v} + \frac{dv}{dt}\hat{\mathbf{t}} + v^2\frac{\hat{\mathbf{n}}}{R} \tag{F.5}$$

Modulo the lack of expansion-shear terms (obviously since we are assuming a rigid reference frame) and factors of γ, the Newtonian formalism (in this form) precisely corresponds to the relativistic analogue (15), or equivalently (17).

In general relativity there does not in general exist extended rigid reference frames, and there cannot be a general analogue of the Newtonian version in the form of (F.1). Indeed we may understand that any instance of $\mathbf{r}'$ should vanish for the general case. Setting $\mathbf{r}' = 0$ in (F.1) we note that we reproduce (F.5) (since then $\mathbf{A_0} = \mathbf{a}_{\text{ref}}$). Thus, as far as it is at all possible for general spacetimes, (15) and (17) conform precisely with the standard Newtonian formalism.

Appendix F.1. A two-step point of view in Newtonian mechanics

Consider in Newtonian mechanics a rigid non-inertial reference frame. For this case we have

$$\mathbf{F}_{\text{aparent}} = \frac{dv}{dt}\hat{\mathbf{t}} + v^2\frac{\hat{\mathbf{n}}}{R} \tag{F.6}$$

Here $\mathbf{F}_{\text{aparent}}$ is the sum of the real and the inertial forces. Relative to the reference frame, henceforth denoted the *base* reference frame, we may choose a new reference frame that may rotate and accelerate relative to the base reference frame. Then we may treat $\mathbf{F}_{\text{aparent}}$ just like we treated the real force $\mathbf{F}$ above, to define a new frame of reference and introduce apparent forces with respect to that frame.

In particular we note that for any particle motion relative to the base reference frame – we can always, as velocity and acceleration are concerned, consider the particle to momentarily move on a circle with accelerating angular velocity. In a rigid coordinate system with origin at the center of the circle in question, and with angular frequency and acceleration to match the particle motion, the particle is at rest and has zero acceleration (momentarily). In these coordinates there is centrifugal force and a transverse (Euler) force whose magnitude and direction are given by $-mv^2\frac{\hat{\mathbf{n}}}{R}$ and $-m\frac{dv}{dt}\hat{\mathbf{t}}$ respectively. These two 'extra' inertial forces will then precisely balance the real and the inertial forces as expressed relative to the base reference frame. Notice however that it is only in this double reference frame sense that it makes sense denote the relative acceleration (multiplied by $-m$) as inertial forces. Note also that by this philosophy, for a rotating base reference frame, we would get two types of centrifugal forces[38].

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[38]Note the difference between the two forces however – the first (standard) centrifugal force can be seen as a *field* – living in the base reference frame independent of the test particle motion. The second is a force defined at a single point and dependent on the motion of the particle. Note also as concerns general relativity – the very name "centrifugal" seems to indicate that there is somewhere a center of some relevance for the motion – in general relativity there can naturally be no such a center for general spacetimes.

For particular applications, such as a static black hole, using a static reference frame, the only inertial force is due to the acceleration of the reference frame – which one may connect in a Newtonian sense to gravity. In standard Newtonian mechanics gravity is not an inertial force, but a real force – hence the original base reference frame has a certain Newtonian inertial flavor to it (modulo curved space, time dilation etc.). From this point of view, the extra reference frame needed to denote the acceleration relative to the base reference frame as inertial forces is "almost" the first reference frame. Likely this philosophy (or something similar) is underlying the ideas by those authors (see e.g. [6, 10]) who denote the terms related to the relative acceleration as inertial forces (when multiplied by $-m$). In this article we are in any case considering just a single reference frame, and are allowing for acceleration relative to that frame.

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