# A covariant formalism of spin precession with respect to a reference congruence

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Submitted 2004-12-10, Published 2005-12-08
Journal Reference: Class. Quantum Grav. **23** 37

**Abstract.** We derive an effectively three-dimensional relativistic spin precession formalism. The formalism is applicable to any spacetime where an arbitrary timelike reference congruence of worldlines is specified. We employ what we call a *stopped* spin vector which is the spin vector that we would get if we momentarily make a pure boost of the spin vector to stop it relative to the congruence. Starting from the Fermi transport equation for the standard spin vector we derive a corresponding transport equation for the stopped spin vector. Employing a spacetime transport equation for a vector along a worldline, corresponding to spatial parallel transport with respect to the congruence, we can write down a precession formula for a gyroscope relative to the local spatial geometry defined by the congruence. This general approach has already been pursued by Jantzen et. al. (see e.g. Jantzen, Carini and Bini 1997 *Ann. Phys.* **215** 1), but the algebraic form of our respective expressions differ. We are also applying the formalism to a novel type of spatial parallel transport introduced in Jonsson (2006 *Class. Quantum Grav.* **23** 1), as well as verifying the validity of the intuitive approach of a forthcoming paper (Jonsson 2007 *Am. Journ. Phys.* **75** 463) where gyroscope precession is explained entirely as a double Thomas type of effect. We also present the resulting formalism in explicit three-dimensional form (using the boldface vector notation), and give examples of applications.

PACS numbers: 04.20.-q, 95.30.Sf

## 1. Introduction

In special and general relativity the spin of a gyroscope is represented by a four-vector $S^\mu$. Assuming that we move the gyroscope without applying any torque to it (in a system comoving with the gyroscope), the spin vector will obey the Fermi transport equation

$$\frac{DS^\mu}{D\tau} = u^\mu \frac{Du^\alpha}{D\tau} S_\alpha. \tag{1}$$

Here $u^\mu$ is the four-velocity of the gyroscope. For a trajectory in a given spacetime, and a spin vector specified at some point along this trajectory, we can integrate (1) to find the spin at any point along the trajectory. The Fermi transport equation is however deceivingly simple since we have not inserted explicitly the affine connection coming from the covariant differentiation. Also, even when we have a flat spacetime and inertial coordinates (so that the affine connection vanishes) the equation is more complex than you might think. As an example we consider motion with fixed speed $v$ along a circle in the $xy$-plane, with an angular frequency $\omega$. Letting the groscope start at $t = 0$ at the positive $x$-axis, we get a set of coupled differential equations

$$\frac{dS^x}{dt} = \gamma^2 v^2 \omega \sin(\omega t)\left(S^x \cos(\omega t) + S^y \sin(\omega t)\right) \tag{2}$$

$$\frac{dS^y}{dt} = -\gamma^2 v^2 \omega \cos(\omega t)\left(S^x \cos(\omega t) + S^y \sin(\omega t)\right) \tag{3}$$

$$\frac{dS^z}{dt} = 0, \qquad S^0 = \mathbf{v} \cdot \mathbf{S} \tag{4}$$

Here $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ and $\mathbf{S}$ is the spatial part of $S^\mu$. For initial conditions $(S^x, S^y, S^z, S^0) = (S, 0, 0, 0)$ the solutions (see [1] p. 175-176) can be written as

$$S^x = S\left(\cos[(\gamma - 1)\omega t] + (\gamma - 1)\sin[\omega\gamma t]\sin[\omega t]\right) \tag{5}$$

$$S^y = S\left(\sin[(1 - \gamma)\omega t] - (\gamma - 1)\sin[\omega\gamma t]\cos[\omega t]\right) \tag{6}$$

$$S^z = 0, \qquad S^0 = -SR\omega\gamma \sin[\omega\gamma t] \tag{7}$$

Looking at $S^x$ and $S^y$, we note that (written in the particular form above) the first terms in respective expression corresponds to a rotation around the $z$-axis, but then there is also another superimposed rotation with time dependent amplitude. To find this solution directly from the coupled differential equations that are the Fermi equations, seems at least at first sight quite difficult, even for this very symmetric and simple scenario.

To get a simpler formalism we may consider, not the spin vector $S^\mu$ itself, but the spin vector we *would* get if we momentarily would stop the gyroscope (relative to a certain inertial frame) by a pure boost (i.e. a non-rotating boost). This object we will call the *stopped* spin vector. While being a four-vector it is effectively a three-dimensional object (having zero time component in the inertial frame in question) and we will show that the spatial part of this object undergoes pure rotation with a constant rate for the example of motion along a circle in special relativity.

Knowing that there is a simple algebraic relation between the stopped and the standard spin vector, the stopped spin vector can be used as an intermediate step to easily find the standard spin vector. There is however also a direct physical meaning to the stopped spin vector, apart from being the spin vector we would get if we stopped the gyroscope. The stopped spin vector *directly* gives the spin as perceived in a comoving system, see section 4.10 for further discussion on this.

In this article we will also consider more general reference frames than inertial ones. For instance we will consider a rotating and accelerating reference frame. This allows us to apply the formalism, via the equivalence principle, to describe in a simple three-dimensional manner how a gyroscope orbiting for instance a rotating black hole will precess relative to the stationary observers. In figure 1 we illustrate how a gyroscope spin vector precesses relative to a vector parallel transported with respect to the spatial geometry.

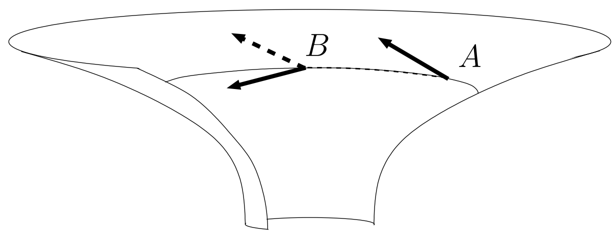


**Figure 1.** A schematic illustration of how an orbiting gyroscope will precess relative to the spatial geometry of a black hole. The full drawn arrow is the stopped spin vector (stopped with respect to the stationary reference observers) of the gyroscope at two different points along the orbit. The dashed arrow is a vector coinciding with the gyroscope spin vector at $A$ and then parallel transported to $B$ with respect to the spatial geometry. For an intuitive explanation of why the gyroscope precesses relative to the spatial geometry even though there are no torques acting on it, see [2].

Given a reference congruence of timelike worldlines, we first derive a general spacetime transport equation for the stopped spin vector (stopped relative to the congruence in question). We then consider a spacetime equation corresponding to *spatial* parallel transport with respect to the spatial geometry defined by the congruence. For the case of a rigid congruence, we easily derive such a transport law. Considering a shearing congruence we use the formalism derived in [3].

Having both the transport equation for the stopped spin vector and the equation for parallel transport, we can put them together and thus get an equation for how fast the stopped spin vector precesses relative to the local spatial geometry connected to the reference congruence. As is the case for the inertial congruence, we will see that the precession corresponds to a simple law of three-rotation.

The general scheme as outlined here has already been pursued by Jantzen et. al. (see [4]), although the angle of approach and the algebraic formalisms are different. The explicit use of the three-dimensional formalism of this paper also appears novel.

This article is complementary to a companion paper [2], where the formalism of relativistic spin precession in three-dimensional language is derived in a very intuitive manner. This paper verifies, through a more formal derivation, the result of [2] for the particular case of a rigid congruence as assumed in [2].

## 2. The stopped spin vector

Let us denote the local four-velocity of our reference congruence by $\eta^\mu$. We introduce a *stopped* spin vector $\bar{S}^\mu$ as the spin vector that we get if we make a pure boost of the spin vector such that it is at rest with respect to the local congruence line. In figure 2 we illustrate in 2+1 dimensions how the two spin vectors are related to each other.

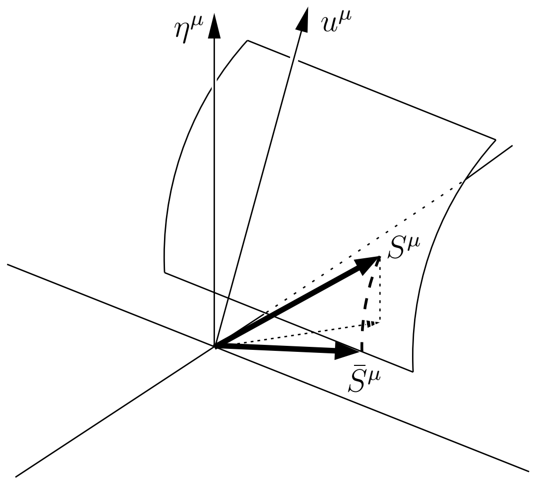


**Figure 2.** A 2+1 illustration of the relation between the spin vector $S^\mu$ and the stopped spin vector $\bar{S}^\mu$. Through the stopping, the tip of the spin vector can in two dimensions be seen as following the hyperbola connected to the Lorentz transformation down to the local slice. Notice that the stopped spin vector is not in general simply the spatial (projected) part of the standard spin vector (the thin dotted arrow).

It follows readily from the Lorentz transformation that we get the stopped vector by removing the $\eta^\mu$-part of $S^\mu$, and shortening the part parallel to the spatial direction of motion by a $\gamma$-factor. Note that the resulting stopped vector is not in general parallel to the spatial part of $S^\mu$. Letting $t^\mu$ be a normalized vector orthogonal to $\eta^\mu$ in the direction of motion, we can express the stopped spin vector as

$$\bar{S}^\mu = \left[\delta^\mu{}_\alpha + \eta^\mu \eta_\alpha + \left(\frac{1}{\gamma} - 1\right) t^\mu t_\alpha\right] S^\alpha. \tag{8}$$

Here we have adopted the spatial sign convention $(-,+,+,+)$ as we will throughout the article. Knowing a little about Thomas precession we may guess that for the simple case of motion along a circle in an inertial frame as discussed earlier, there is a simple law of three-dimensional rotation for this object. Indeed in the following discussion we will show this, and at the same time consider the effects of rotation coming from having non-inertial reference frames (connected to $\eta^\mu$).

We also need an explicit expression for the standard spin vector in terms of the stopped spin-vector $\bar{S}^\mu$. The relationship between the two vectors follows readily from the Lorentz-transformation:

$$S^\mu = \bar{S}^\alpha K^\mu{}_\alpha \tag{9}$$

$$K^\mu{}_\alpha = \left[\delta^\mu{}_\alpha + \gamma v \eta^\mu t_\alpha + (\gamma - 1) t^\mu t_\alpha\right]. \tag{10}$$

This we may now insert into the Fermi transport equation to derive an expression for the *stopped* spin vector.

## 3. Covariant derivation of the transport equation for the stopped spin-vector

In this section we consider gyroscope transport relative to an arbitrary reference congruence $\eta^\mu$. For a spin vector $S^\mu$ transported along a worldline of four-velocity $u^\mu$, we have the Fermi transport law

$$\frac{DS^\mu}{D\tau} = u^\mu S^\rho \frac{Du_\rho}{D\tau}. \tag{11}$$

Using (9) in (11) readily yields

$$\frac{D\bar{S}^\alpha}{D\tau} K^\mu{}_\alpha = \bar{S}^\alpha \left[ u^\mu K^\rho{}_\alpha \frac{Du_\rho}{D\tau} - \frac{DK^\mu{}_\alpha}{D\tau} \right]. \tag{12}$$

We need now the inverse of $K^\mu{}_\alpha$ to get an explicit transport equation for the stopped spin vector. Through a general ansatz[1], we find

$$K^{-1\nu}{}_\mu = \delta^\nu{}_\mu + \left( \frac{1}{\gamma} - 1 \right) t^\nu t_\mu - v\eta^\nu t_\mu. \tag{13}$$

That this is indeed the inverse of $K^\mu{}_\alpha$ is easy to verify[2]. So we have

$$\frac{D\bar{S}^\nu}{D\tau} = \bar{S}^\alpha \left[ u^\mu K^\rho{}_\alpha \frac{Du_\rho}{D\tau} - \frac{DK^\mu{}_\alpha}{D\tau} \right] K^{-1\nu}{}_\mu. \tag{14}$$

Here we have the desired expression. In Appendix A we expand and simplify this to find

$$\begin{aligned} \frac{D\bar{S}^\mu}{D\tau} = {} & \frac{\gamma v}{\gamma + 1} \bar{S}^\alpha \left( t^\mu \left[ \frac{D}{D\tau} (u_\alpha + \eta_\alpha) \right]_\perp - t_\alpha \left[ \frac{D}{D\tau} (u^\mu + \eta^\mu) \right]_\perp \right) \\ & + \eta^\mu \bar{S}^\alpha \frac{D\eta_\alpha}{D\tau}. \end{aligned} \tag{15}$$

By the perpendicular sign $\perp$ we here mean that we should select only the part orthogonal to both $t^\mu$ and $\eta^\mu$. Note that $\frac{D}{D\tau}$ means covariant differentiation along the gyroscope worldline. Equation (15) then tells us how the stopped spin vector deviates from a parallel transported vector relative to a freely falling system. In fact we notice from the antisymmetric form of (15) that (excepting the $\eta^\mu$ term) it corresponds to a spatial rotation (see section 4.2 for a more detailed argument). That seems very reasonable since it insures that the norm of the stopped spin vector will be constant (consider the rotation with respect to a freely falling system locally comoving with the congruence). We also see that only if $u^\mu + \eta^\mu$ changes along the gyroscope worldline, with respect to a freely falling system, do we get a net rotation relative to this freely falling system.

Introducing the wedge product defined by $a^\alpha \wedge b^\beta \equiv a^\alpha b^\beta - b^\alpha a^\beta$ and the projection operator $P^\mu{}_\alpha = \delta^\mu{}_\alpha + \eta^\mu \eta_\alpha$, we can put (15) in a more compact form

$$P^\mu{}_\alpha \frac{D\bar{S}^\alpha}{D\tau} = \frac{\gamma v}{\gamma + 1} \bar{S}^\alpha \left( t^\mu \wedge \left[ \frac{D}{D\tau} (u_\alpha + \eta_\alpha) \right]_\perp \right). \tag{16}$$

[1] We have $K^{-1\nu}{}_\rho K^\rho{}_\alpha = \delta^\nu{}_\alpha$. The ansatz is of the form $K^{-1\nu}{}_\alpha = \delta^\nu{}_\alpha + a t^\mu t_\alpha + b t^\mu \eta_\alpha + c\eta^\mu t_\alpha + d\eta^\mu \eta_\alpha$.

[2] In defining $K^\mu{}_\alpha$ we are free to add terms containing $\eta_\alpha$, since these anyway die when multiplied by $\bar{S}^\alpha$. If we instead would have defined $K^\mu{}_\alpha = \delta^\mu{}_\alpha + \frac{1}{\gamma+1}(u^\mu + \eta^\mu)(u_\alpha - \eta_\alpha)$ we would get the inverse $K^{-1\mu}{}_\alpha = \delta^\mu{}_\alpha + \frac{1}{\gamma+1}(u^\mu + \eta^\mu)(\eta_\alpha - u_\alpha)$ . Here the perfect symmetry in $S^\mu, \eta^\mu$ and $\bar{S}^\mu, u^\mu$ is transparent. There however does not appear to be any particular advantages of this gauge.

Incidentally we may note that, as regards $t^\mu$-components within the bracketed expression, we do not need the $\perp$ sign. Any $t^\mu$ components within the bracketed expressions will cancel due to the anti-symmetrization as is easy to see. We however keep the $\perp$ sign to indicate orthogonality to $\eta^\mu$. The simple form of (16) appears to be novel.

## 4. Application to flat spacetime, and inertial congruences

While we have yet to put the formalism in its final form, some applications and discussion may be useful already at this point for the simple case of an inertial reference congruence in special relativity.

### *4.1. Employing the spatial curvature of the gyroscope trajectory*

As a particular example, consider a flat spacetime with an inertial congruence. For this case it is not hard to show, see e.g [3], that the spatial curvature of a trajectory depends on the four-acceleration as

$$\left[\frac{Du_\alpha}{D\tau}\right]_\perp = \gamma^2 v^2 \frac{n_\alpha}{R}. \tag{17}$$

Here $R$ is the spatial curvature[3] and $n^\mu$ is a normalized four-vector, orthogonal to the inertial congruence $\eta^\mu$, pointing in the direction of spatial curvature. Using this in (16) we get

$$P^\mu{}_\alpha \frac{D\bar{S}^\alpha}{D\tau} = \gamma v(\gamma - 1)\bar{S}^\alpha \left(t^\mu \wedge \frac{n_\alpha}{R}\right). \tag{18}$$

As we will see in the following section, this differential equation corresponds to a three-dimensional rotation.

### *4.2. Three-dimensional formalism, for flat spacetime and an inertial congruence*

Choosing inertial coordinates adapted to the inertial congruence in question so that $\bar{S}^\mu = (0, \bar{\mathbf{S}})$, $t^\mu = (0, \hat{\mathbf{t}})$ and $n^\mu = (0, \hat{\mathbf{n}})$ we get from (18)

$$\frac{d\bar{\mathbf{S}}}{d\tau} = \gamma v(\gamma - 1)\left[\hat{\mathbf{t}}(\bar{\mathbf{S}} \cdot \frac{\hat{\mathbf{n}}}{R}) - \frac{\hat{\mathbf{n}}}{R}(\bar{\mathbf{S}} \cdot \hat{\mathbf{t}})\right]. \tag{19}$$

The expression within the brackets is a vector triple product and we may write it as a double cross product. Letting $\mathbf{v} = v\hat{\mathbf{t}}$ we get

$$\frac{d\bar{\mathbf{S}}}{d\tau} = \gamma(\gamma - 1)\left(\frac{\hat{\mathbf{n}}}{R} \times \mathbf{v}\right) \times \bar{\mathbf{S}}. \tag{20}$$

[3] As is illustrated in [3] there are plenty of ways to define spatial curvature measures in general, but for an inertial congruence most of these coincide with the standard projected curvature that we here assume.

Rather than using $\tau$ we could use local time $\tau_0$ (the time as experienced by observers at rest relative to the inertial congruence in question) in which case we get a gamma factor less on the right hand side.

$$\frac{d\bar{\mathbf{S}}}{d\tau_0} = (\gamma - 1)\left(\frac{\hat{\mathbf{n}}}{R} \times \mathbf{v}\right) \times \bar{\mathbf{S}}. \tag{21}$$

This is the famous Thomas precession, in stopped spin vector three-formalism. Introducing $\mathbf{\Omega}$ as the precession vector, around which the stopped spin vector rotates, we can alternatively write (21) as

$$\frac{d\bar{\mathbf{S}}}{d\tau_0} = \mathbf{\Omega} \times \bar{\mathbf{S}} \tag{22}$$

$$\mathbf{\Omega} = (\gamma - 1)\left(\frac{\hat{\mathbf{n}}}{R} \times \mathbf{v}\right). \tag{23}$$

Looking at (22) component-wise, it is a set of coupled differential equations, just like the standard Fermi equations. Unlike the Fermi-equations however, the new equations correspond to a simple law of rotation (precession).

### *4.3. The circular motion revisited*

As a specific example we may consider, as in the introduction, the precession of a gyroscope transported at constant speed $v$ around a circle of radius $R$ in the $z = 0$ plane. Assuming a motion with a clockwise angular velocity $\omega = v/R$, the counterclockwise angular velocity $\Omega$ for the precession of the stopped spin vector is then according to (23) given by

$$\Omega = (\gamma - 1)\omega. \tag{24}$$

Consider then for instance the net precession after one lap. The local time per lap is simply $2\pi/\omega$ and hence the net precession angle (in radians) around the plane normal is given by $2\pi(\gamma - 1)$. If the circular motion is counter-clockwise, the precession is clockwise and vise versa.

### *4.4. Re-deriving the solution for the standard spin vector*

We can also trivially find the solution for the standard (projected) spin vector for the case of circular motion with constant speed with initial conditions as listed in the example in the introduction. We know that the standard (projected) spin vector is related to the stopped spin vector through a lengthening of the stopped spin vector in the forward direction of motion $\hat{\mathbf{t}}$ by a $\gamma$-factor. We have then

$$\mathbf{S} = \bar{\mathbf{S}} + (\gamma - 1)(\bar{\mathbf{S}} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}}. \tag{25}$$

Using the notation of the previous subsection we have then trivially for the case at hand

$$\bar{\mathbf{S}} = S\cos(\Omega t)\hat{\mathbf{x}} - S\sin(\Omega t)\hat{\mathbf{y}} \tag{26}$$

$$\hat{\mathbf{t}} = -\sin(\omega t)\hat{\mathbf{x}} + \cos(\omega t)\hat{\mathbf{y}}. \tag{27}$$

Using these expressions in (25) we immediately get the desired solution. Using elementary rules for manipulating the trigonometric functions we can write it in the form of (5)-(7). If we are interested also in $S^0$, it is given by the orthogonality of the standard spin vector and the four-velocity as $S^0 = \mathbf{S}\cdot\mathbf{v}$. Note that by use of the stopped spin vector formalism there is effectively no differential equation solving involved for this simple case.

### *4.5. A special relativistic theorem of spin precession for planar constant velocity motion*

For motion in a circle with constant velocity, the Fermi equation can be solved without use of the stopped spin vector formalism, although the solution is a bit complicated. What about if we consider motion with constant velocity along some other curve, say a part of a parabola or some more irregular curve? Then the Fermi equation would likely appear to be very complicated to solve analytically in the general case. Using the method with the stopped spin vector the solution can however trivially be found for arbitrary curves. First let us state a small theorem that we will then easily prove.

> *The stopped spin vector of a gyroscope transported with constant speed $v$ along a smooth curve in a spatial plane in a flat spacetime will rotate a net clockwise angle around the normal of the plane given by $\Delta\alpha_{precess} = (\gamma - 1)\Delta\alpha_{curve}$ where $\Delta\alpha_{curve}$ is the net counterclockwise turning angle of the tangent direction of the curve.*

Note that the parameter $\Delta\alpha_{\text{curve}}$ may be larger than $2\pi$. For a simple closed curve (one that is not crossing itself), assuming the gyroscope to be transported once around the curve, we have $\Delta\alpha_{\text{curve}} = 2\pi$.

This theorem is easily proven by dividing an arbitrary smooth curve into infinitesimal segments within which we can consider the local curvature radius to be constant. Letting $\omega$ denote the counter-clockwise angular velocity of the forward direction of motion $\hat{\mathbf{t}}$ (so $\omega = d\alpha_{\text{curve}}/dt$), we have according to (24) the clockwise angular velocity as $\Omega = (\gamma - 1)\omega$. Thus the net angles of the gyroscope precession and the turning of the forward direction of the curve, along the segment in question, are related through $d\alpha_{\text{precess}} = (\gamma - 1)d\alpha_{\text{curve}}$. Adding up the precession contributions from all the segments of the curve we get

$$\Delta\alpha_{\text{precess}} = (\gamma - 1)\Delta\alpha_{\text{curve}}. \tag{28}$$

Thus the theorem is proven. Note that while the motion is assumed to be in a plane, the spin vector may point off the plane.

### *4.6. Some consequences of the theorem*

We can draw a conclusion from the above proven theorem (also knowing that there is a simple algebraic relation between the stopped and the standard spin-vector) that can be expressed in terms of the standard spin vector, without reference to the stopped spin vector. Consider then a smooth simple closed curve and let a certain point along this curve be the initial position for the gyroscope. For given initial spin vector, initial direction of motion[4] and constant speed $v$, the final spin vector (after one lap around the loop) is independent of the shape of the loop as illustrated in figure 3.

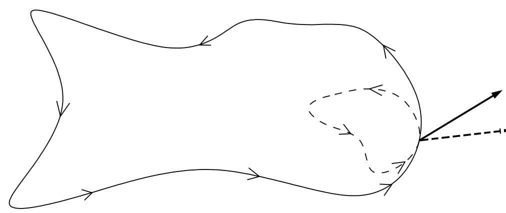


**Figure 3.** Illustrating that for a fixed initial direction of motion, fixed initial spin vector, and fixed constant speed $v$ – the final spin vector after one lap around any simple smooth closed curve is independent of the shape of the curve.

But of course, the theorem is stronger than this. Given an arbitrary, not necessarily closed but smooth curve along which we transport the gyroscope with constant velocity, we can trivially find the standard spin vector at any point along the curve. We take the spatial part of the initial spin vector and shorten the part parallel to the direction of motion by a $\gamma$-factor to form the initial stopped spin vector. For any given curve $\mathbf{x}(\lambda)$ we then calculate the initial direction of the curve together with the direction of the curve at the point in question. Then, modulo a winding number times $2\pi$[5], we can trivially find the corresponding $\Delta\alpha_{\text{curve}}$ and thus through (28) the corresponding stopped spin vector at the point in question. Lengthening the parallel part of the stopped spin vector by a factor $\gamma$, we get the spatial part of the standard spin vector at the point in question. If we are interested in the zeroth component of the standard spin vector it is given by $S^0 = \mathbf{S} \cdot \mathbf{v}$. Thus solving a possibly very complicated differential equation is reduced to performing a few algebraic steps[6].

### *4.7. More complicated motion*

For motion in a plane where the velocity is not constant, the procedure is analogous to that described in section 4.6 except that we need to integrate (a single integral which may or may not be complicated to solve analytically) to find $d\alpha_{\text{precess}}$. For the most general motion, not necessarily confined to a plane and with a speed that may vary, it

[4]One cannot in general keep the standard (unlike the stopped) spin vector fixed while altering the initial direction of motion of the gyroscope since the standard spin vector must be orthogonal to the gyroscope four-velocity.

[5]The only non-trivial part of calculating the turning angle lies in finding out the number of turns taken by the curve since for a curve $\mathbf{x}(\lambda)$ we only get the turning angle $\Delta\alpha_{\text{curve}}$ up to a term $2\pi n$, where $n$ is an integer, from the local quantity $\frac{d\mathbf{x}}{d\lambda}$

[6]Again modulo the winding number mentioned earlier. For many cases, like for instance for a parabola, this however presents no problem at all.

is however not just a matter of ordinary integration[7]. Given an arbitrary motion $\mathbf{x}(\tau_0)$ along a smooth curve we can however solve a differential equation, given by (22) and (23), for $\bar{\mathbf{S}}$. Likely this differential equation will be simpler to solve than the Fermi equation.

### *4.8. A comment on the relation between the intrinsic angular momentum, the projected spin vector, the gyroscope axis and the stopped spin vector*

To gain further intuition on the meaning of the stopped spin vector it may be useful to explore how it is related to other vectors of physical interest connected to the gyroscope spin. In particular we may consider the gyroscope intrinsic angular momentum, and the momentary direction of the gyroscope axis as perceived in the reference system in question (where the observers are integral curves of $\eta^\mu$).

Consider then a gyroscope moving along a straight line in the $xy$-plane in special relativity (using inertial coordinates) with constant speed. The gyroscope axis is assumed to lie in the plane of motion and to be tilted somewhere between the forward and the sideways direction. In 2+1 dimensions we can easily visualize the worldsheet of the gyroscope central axis as well as various vectors of interest, see figure 4.

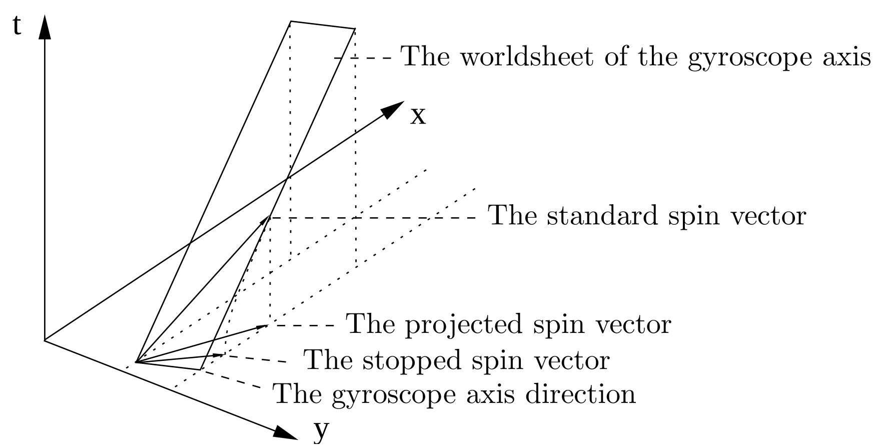


**Figure 4.** A sketch in 2+1 dimensions of vectors related to a spinning gyroscope.

We note that there are (at least) three different spatial directions of relevance for the gyroscope. It is easy to realize (length contraction) that the direction of the gyroscope axis is simply related to the direction of the stopped spin vector through a

[7]One could for instance represent a finite precession (rotation) by a vector whose direction determines the axis of rotation and whose norm determines the angle (in radians) of the precession. It is however easy to realize that for a a finite such rotation (like the net rotation after some finite stretch along a trajectory) followed by an infinitesimal rotation around some other axis – one cannot in general simply add the two corresponding rotation vectors (to first order) to form a new rotation vector. Of course there are examples of non-planar motion, like motion along a helix for instance, where the precession vector remains in the same direction for which case it is a simple matter of integration after all to find the net rotation of the stopped spin vector.

gamma factor. Given any of these directions the other two can thus easily be found. Furthermore one can show, at least for an idealized scenario as considered in Appendix B, that the the intrinsic angular momentum, that we will denote $\mathbf{S}_{\mathrm{L}}$, is in fact given by $\mathbf{S}/\gamma$. The various vectors involved are illustrated in figure 5.

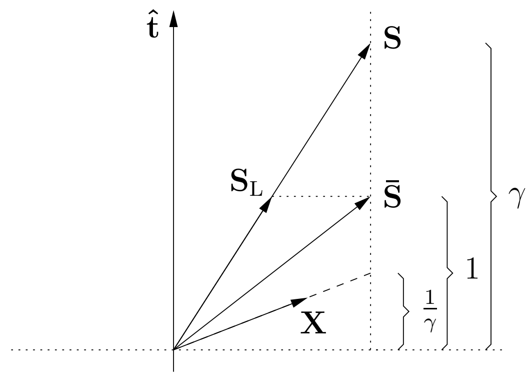


**Figure 5.** The three different directions in question are simply related through a stretching by a gamma factor in the direction of motion. In this illustration a gamma factor of 2 was assumed, with motion in the upwards direction ($\hat{\mathbf{t}}$). Note that the depicted norm of the gyroscope axis vector $\mathbf{X}$ is arbitrary.

### *4.9. Four vectors, four differential equations*

Consider a spatial vector $\mathbf{X}$ that connects the base of the gyroscope to the tip of the gyroscope, as perceived in the reference system connected to $\eta^\mu$. We understand that this vector evolves according to a simple rule of rotation given by (21) modulated by a contraction by a factor of $\gamma$ in the direction of motion. It is a short exercise to show that this means that $\mathbf{X}$ in fact obeys a rather compact differential equation

$$\frac{d\mathbf{X}}{d\tau_0} = -\gamma^2 \frac{d\mathbf{v}}{d\tau_0} \left[\mathbf{X} \cdot \mathbf{v}\right]. \tag{29}$$

We can perform a corresponding analysis for the projected spin vector to find[8]

$$\frac{d\mathbf{S}}{d\tau_0} = \gamma^2 \mathbf{v} \left[\mathbf{S} \cdot \frac{d\mathbf{v}}{d\tau_0}\right]. \tag{30}$$

The equations for the stopped spin vector can be written in the form

$$\frac{d\bar{\mathbf{S}}}{d\tau_0} = \frac{\gamma - 1}{v^2} \left(\frac{d\mathbf{v}}{d\tau_0} \times \mathbf{v}\right) \times \bar{\mathbf{S}}. \tag{31}$$

From (30), letting $\mathbf{S} = \gamma \mathbf{S}_{\mathrm{L}}$, we readily find

$$\frac{d\mathbf{S}_{\mathrm{L}}}{d\tau_0} = \gamma^2 \mathbf{v} \left[\mathbf{S}_{\mathrm{L}} \cdot \frac{d\mathbf{v}}{d\tau_0}\right] - \gamma^2 v \frac{dv}{dt} \mathbf{S}_{\mathrm{L}}. \tag{32}$$

[8]This also follows readily from the standard Fermi equations for the case of inertial coordinates in special relativity.

Comparing the four differential equations we see that they are all quite compact, although the equation for the stopped spin vector, corresponding to a pure rotation, is more likely to be simple to solve (as we have seen for the example of motion on a circle).

### *4.10. A comment on the meaning and purpose of the stopped spin vector*

One might argue that the object of physical interest is the intrinsic (spin) angular momentum of the gyroscope which is given by $\mathbf{S}/\gamma$, or perhaps the observed direction of the gyroscope central axis. From this point of view the stopped spin vector is in a sense a means to an end. By using the stopped spin vector as an intermediate step we can find the solutions to otherwise quite complicated differential equations for the objects of physical interest. From a mathematical point of view this is certainly sufficient to motivate the use of the stopped spin vector. There is however more to the stopped spin vector than this. In particular we note that the stopped spin vector *directly* gives us the spin as perceived in a comoving system. For instance, if the stopped spin vector is at a $45°$ angle with respect to the forward direction – so it will be with respect to a system comoving with the gyroscope[9]. This is contrary to the standard spin vector which only gives the spin direction with respect to the comoving system after a Lorentz transformation. Consider the following example. A gyroscope is suspended inside a satellite such that no torque is exerted on the gyroscope as seen from the satellite. The satellite is assumed to be orbiting along some predetermined smooth simple closed curve, on a plane in special relativity[10], using it's jet engines to stay on the path. Suppose then that we wish to measure, from the satellite, the precession angle of the gyroscope (as predicted by relativity) after a full orbit (or maybe several full orbits). We note that the direction of the gyroscope relative to the satellite itself is not a good measure[11]. Assuming that we have a couple of fixed stars, we can however use the direction of these stars (as perceived from the satellite) as guidelines to set up a reference system within the satellite[12] For this scenario the stopped spin vector is exactly the physical object that we are interested in. It exactly represents the gyroscope direction relative to the star-calibrated reference system of the satellite. Thus if the stopped spin vector turns a certain angle, that is precisely the turning angle of the gyroscope relative to the

[9]If the stopped spin vector has certain components with respect to a set of base vectors adapted to the reference congruence in question, then those components precisely corresponds to the components of the standard spin vector with respect to a boosted version (a pure boost to comove with the gyroscope) of the base vectors just mentioned. This viewpoint is mentioned in [1] p. 1117, although they do not consider general spacetimes and velocities.

[10]The general argument works also for gyroscopes orbiting the earth in a general relativistic treatment. More on this in section 10.

[11]The satellite may have had an initial rotation from the start or the jet-engines may give it one. Also, even if it would have zero proper rotation then the gyroscope would keep its direction relative to the satellite and thus would not turn at all relative to the satellite.

[12]We also assume that the satellite has some way of knowing when it is at its initial position (so it knows when to calibrate its coordinates with respect to the stars).

star-calibrated reference system of the satellite.

While we are here focusing on spinning gyroscopes, it should also be noted that the formalism of the stopped spin vector is immediately applicable to describe the resulting rotation of *any* object which has zero *proper* (comoving) rotation.

In conclusion, the stopped spin vector may be used as an intermediate step to simplify the calculation of the evolution of the intrinsic angular momentum (spin) of a gyroscope, or the perceived direction of the gyroscope axis. The stopped spin vector is however also of direct physical importance since it gives us the spin as perceived in a comoving system.

So far we have only given examples that apply to flat spacetime, and inertial reference frames. As we will see in the following sections the stopped spin vector can be just as useful also for curved spacetimes and non-inertial reference frames.

## 5. Spatial parallel transport

The transport equation (16) tells us how the stopped spin vector deviates from a vector that is parallel transported with respect to the spacetime geometry. This by itself is however not really what we are after if the reference congruence is non-inertial. To get a truly three-dimensional formalism, we in stead want an expression telling us how fast the stopped spin vector deviates (rotates) from a vector that is parallel transported with respect to the *spatial* geometry determined by the congruence. As is demonstrated in [4] and in [3], it is possible derive a *spacetime* transport law corresponding to a *spatial* parallel transport. For the simple, and perhaps most useful, case of a rigid congruence[13] the issue is sufficiently simple that we will briefly review it in the coming subsection.

### *5.1. Rigid congruence*

Suppose then that we have a rigid congruence with nonzero acceleration $a^\mu$, nonzero rotation tensor $\omega^\mu{}_\nu$ but with vanishing expansion-shear tensor $\theta^\mu{}_\nu$[14].

In figure 6 we show an illustration of the spacetime transport of a vector orthogonal to the congruence.

It is easy to show that in the coordinates of a freely falling system $(t, x^k)$, locally comoving with the congruence, the velocity of the congruence points (assuming vanishing $\theta^\mu{}_\nu$) is to first order given by

$$v^k = \omega^k{}_j x^j + a^k t. \tag{33}$$

Knowing that the velocity of the congruence is zero to lowest order, relative to the inertial system in question, we need not worry about length contraction and such. It

[13] The congruence may rotate and accelerate but it may not shear or expand.

[14] The kinematical invariants of the congruence are defined as (see [1] p. 566): The expansion scalar $\theta = \nabla_\alpha \eta^\alpha$, the acceleration vector $a^\mu = \eta^\alpha \nabla_\alpha \eta^\mu$, the shear tensor $\sigma_{\mu\nu} = \frac{1}{2}\left(\nabla_\rho \eta_\mu P^\rho{}_\nu + \nabla_\rho \eta_\nu P^\rho{}_\mu\right) - \frac{1}{3}\theta P_{\mu\nu}$ and the rotation tensor $\omega_{\mu\nu} = \frac{1}{2}\left(P^\rho{}_\nu \nabla_\rho \eta_\mu - P^\rho{}_\mu \nabla_\rho \eta_\nu\right)$. Furthermore we employ what we denote the expansion-shear tensor $\theta_{\mu\nu} = \frac{1}{2}\left(P^\rho{}_\nu \nabla_\rho \eta_\mu + P^\rho{}_\mu \nabla_\rho \eta_\nu\right)$.

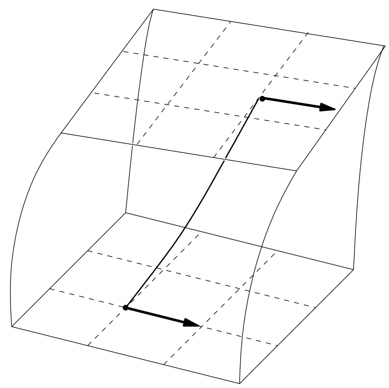

**Figure 6.** A 2+1 illustration of transporting a spatial vector along a worldline, seen from freely falling coordinates locally comoving with the congruence. As the reference coordinates rotate due to $\omega^{\mu}{}_{\alpha}$, so should the vector in order for it to be proper spatially transported.

is then easy to realize that the proper spacetime transport law of a spatial vector $k^{\mu}$ corresponding to standard spatial parallel transport is

$$\frac{Dk^{\mu}}{D\tau} = \gamma\omega^{\mu}{}_{\alpha}k^{\alpha} + b\eta^{\mu}. \tag{34}$$

Here $b$ can easily be determined from the orthogonality of $k^{\mu}$ and $\eta^{\mu}$[15]. Here we have then a spacetime transport equation corresponding to spatial parallel transport, for the case of a non-shearing (non-expanding) congruence.

### *5.2. Including shear and expansion*

For a more complicated congruence that is shearing and expanding, it is not quite so obvious how to define the spatial parallel transport. Indeed as discussed in e.g [4] and [3], there are several ways of doing this. We will here follow the approach of [3], and consider two different such parallel transports. These transports are connected to two different ways of defining a spatial curvature for a test particle worldline, with respect to the congruence

$$\text{Projected:} \quad \frac{1}{\gamma^2}\left[\frac{Du^{\mu}}{D\tau}\right]_{\perp} = [a^{\mu}]_{\perp} + 2v(\omega^{\mu}{}_{\alpha}t^{\alpha} + [\theta^{\mu}{}_{\alpha}t^{\alpha}]_{\perp}) + v^2\frac{n^{\mu}_{\mathrm{ps}}}{R_{\mathrm{ps}}} \tag{35}$$

$$\text{New:} \quad \frac{1}{\gamma^2}\left[\frac{Du^{\mu}}{D\tau}\right]_{\perp} = [a^{\mu}]_{\perp} + 2v\omega^{\mu}{}_{\alpha}t^{\alpha} + v^2\frac{n^{\mu}_{\mathrm{ns}}}{R_{\mathrm{ns}}}. \tag{36}$$

Here $R_{\mathrm{ps}}$ and $n^{\mu}_{\mathrm{ps}}$ are the curvature and the curvature direction that we get if we project the the spacetime trajectory down along the congruence onto a local timeslice (orthogonal to the congruence at the point in question). The suffix 'ps' stands for 'Projected Straight'. The curvature $R_{\mathrm{ns}}$ and the curvature direction $n^{\mu}_{\mathrm{ns}}$ are defined

[15] From the orthogonality $k^{\mu}\eta_{\mu} = 0$ follows (differentiate $\frac{D}{D\tau}$ along the gyroscope worldline) that $\frac{Dk^{\mu}}{D\tau}\eta_{\mu} = -k^{\mu}\frac{D\eta_{\mu}}{D\tau}$. Contracting both sides of (34) by $\eta_{\mu}$ gives $b = k^{\mu}\frac{D\eta_{\mu}}{D\tau}$.

with respect to deviations from a certain (new) notion of a spatially straight line. The latter is defined as a line that with respect to variations in the projected curvature, leaves the integrated spatial distance (as defined by the congruence) unaltered (to first order in the variation). As it turns out, a straight line with respect to this definition, has in general a non-zero projected curvature when the congruence is shearing. The suffix 'ns' stands for 'New-Straight'. This particular curvature is connected to Fermat's principle, and optical geometry [3, 5].

For brevity we let the suffix 's' denote either 'ps', or 'ns'. Introducing $C_{\mathrm{ps}} = 1$, $C_{\mathrm{ns}} = 0$ we can then express both curvatures jointly as

$$\frac{1}{\gamma^2}\left[\frac{Du^\mu}{D\tau}\right]_\perp = [a^\mu]_\perp + 2v(\omega^\mu{}_\alpha t^\alpha + C_{\mathrm{s}}[\theta^\mu{}_\alpha t^\alpha]_\perp) + v^2\frac{n^\mu_{\mathrm{s}}}{R_{\mathrm{s}}}. \tag{37}$$

From these two curvature measures one can introduce corresponding equations for spatial parallel transports [3]. A joint expression for the parallel transport of a vector $k^\mu$ is given by

$$\frac{Dk^\mu}{D\tau} = \gamma k^\alpha \omega^\mu{}_\alpha + \gamma(2C_{\mathrm{s}} - 1)k^\alpha(\theta^\mu{}_\beta t^\beta \wedge t_\alpha) + \eta^\mu k^\alpha \frac{D\eta_\alpha}{D\tau}. \tag{38}$$

Here $\frac{D\eta_\alpha}{D\tau}$ is the covariant derivative along the (gyroscope) worldline in question. Notice that for vanishing shear expansion tensor, the two transports both correspond to (34).

Having defined two types of parallel transport according to (38), we can define corresponding covariant differentiations along a curve as

$$\frac{D_{\mathrm{s}}k^\mu}{D_{\mathrm{s}}\tau} = \frac{Dk^\mu}{D\tau} - \gamma k^\alpha\left(\omega^\mu{}_\alpha + (2C_{\mathrm{s}} - 1)(\theta^\mu{}_\beta t^\beta \wedge t_\alpha)\right) - \eta^\mu k^\alpha \frac{D\eta_\alpha}{D\tau}. \tag{39}$$

These derivatives then tells us how fast a vector deviates from a corresponding parallel transported vector (momentarily parallel to the vector in question). Substituting $k^\mu \to \bar{S}^\mu$ and using (16) we get the equations for how fast the stopped spin vector precesses relative to a spatially parallel transported vector (of the two types). First we however rewrite (16).

## 6. Rewriting the stopped spin vector transport equation

We saw in the preceding section how the kinematical invariants of the congruence entered naturally in the definition of spatial parallel transport. We can also expand $(\frac{D\eta_\alpha}{D\tau} + \frac{Du_\alpha}{D\tau})$, in the transport equation (16) for the stopped spin vector, in terms of the kinematical invariants of the congruence. First of all we have

$$\frac{D\eta_\alpha}{D\tau} = u^\rho \nabla_\rho \eta_\alpha = \gamma(\eta^\rho + vt^\rho)\nabla_\rho \eta_\alpha. \tag{40}$$

Also we know that (see e.g [1] p. 566)

$$\nabla_\rho \eta_\alpha = \omega_{\alpha\rho} + \theta_{\alpha\rho} - a_\alpha \eta_\rho. \tag{41}$$

Using (40), we have then

$$\frac{D\eta_\alpha}{D\tau} = \gamma v\left(\omega_{\alpha\rho}t^\rho + \theta_{\alpha\rho}t^\rho\right) + \gamma a_\alpha. \tag{42}$$

Using this together with (37) in (16), also adding the proper $\eta^\mu$-term enabling the removal of the projection operator in (16), we readily find

$$\frac{D\bar{S}^\mu}{D\tau} = \frac{\gamma v}{\gamma+1}\bar{S}^\alpha t^\mu \wedge \left[\gamma(\gamma+1)a_\alpha + \gamma v(2\gamma+1)\omega_{\alpha\rho}t^\rho \right.$$
$$\left. + \gamma v(2\gamma C_{\mathrm{s}}+1)\theta_{\alpha\rho}t^\rho + \gamma^2 v^2 \frac{n_{\mathrm{s}\alpha}}{R_{\mathrm{s}}}\right] + \eta^\mu \bar{S}^\alpha \frac{D\eta_\alpha}{D\tau}. \quad (43)$$

Notice that we have omitted the perpendicular signs ($\perp$) on $\theta_{\alpha\rho}t^\rho$ and $a_\alpha$ since these objects are already orthogonal to $\eta^\mu$ and any $t^\mu$ components die due to the anti-symmetrization.

## 7. The rotation of the stopped spin vector relative to a parallel transported vector

Now it is time to put together the results of the preceding two sections. What we want is the net rotation of the stopped spin vector relative to a spatially parallel transported vector. Using (43) and (39) (setting $k^\alpha = \bar{S}^\alpha$), we then readily find

$$\frac{D_{\mathrm{s}}\bar{S}^\mu}{D_{\mathrm{s}}\tau} = \bar{S}^\alpha \left[\gamma^2 v(t^\mu \wedge a_\alpha) + (\gamma-1)(2\gamma+1)(t^\mu \wedge \omega_{\alpha\rho}t^\rho) - \gamma\omega^\mu{}_\alpha \right.$$
$$\left. + (2\gamma^2 C_{\mathrm{s}}-1)(t^\mu \wedge \theta_{\alpha\rho}t^\rho) + \gamma v(\gamma-1)\left(t^\mu \wedge \frac{n_{\mathrm{s}\alpha}}{R_{\mathrm{s}}}\right)\right]. \quad (44)$$

Here $C_{\mathrm{ps}} = 1$ and $C_{\mathrm{ns}} = 0$. So this gives us how fast a gyroscope stopped spin vector deviates from a corresponding (spatially) parallel transported vector. In particular considering the expression in a freely falling system locally comoving with the congruence, we understand that the expression within the brackets on the right hand side is simply the effective rotation tensor relative to the spatial geometry.

It could be practical with an expression corresponding to (44) but where the proper four-acceleration is explicit. Using (16), (39) and (42) we readily find

$$\frac{D_{\mathrm{s}}\bar{S}^\mu}{D_{\mathrm{s}}\tau} = \frac{\gamma v}{\gamma+1}\bar{S}^\alpha t^\mu \wedge \left[\left[\frac{Du_\alpha}{D\tau}\right]_\perp + \gamma a_\alpha + \gamma v\omega_{\alpha\rho}t^\rho + (2\gamma C_{\mathrm{s}}-1)\frac{\gamma+1}{\gamma v}\theta_{\alpha\rho}t^\rho\right]$$
$$- \gamma\omega^\mu{}_\alpha \bar{S}^\alpha. \quad (45)$$

Notice that the expression for the four-acceleration here (naturally) is independent of what curvature measure that we use. Still (45) depends on what curvature measure we are using (manifesting itself in the occurrence of $C_{\mathrm{s}}$) assuming non-zero $\theta_{\alpha\rho}t^\rho$, since the transport laws for the two types of spatial parallel transport differs.

## 8. Three-dimensional formalism, assuming rigid congruence

We can rewrite (44) and (45) as purely three-dimensional equations. For any specific global labeling of the congruence lines (i.e. any specific set of spatial coordinates adapted to the congruence) we can locally choose a time slice orthogonal to the congruence so that $\bar{S}^\mu = (0, \bar{\mathbf{S}})$. This then uniquely defines the three-vector $\bar{\mathbf{S}}$ at any point along the gyroscope trajectory. Analogous to what we did in going from (18) to (20), for a set of vectors $\bar{S}^\mu$, $t^\mu$ and $k^\mu$ orthogonal to the congruence, we let $\bar{S}^\alpha t^\mu \wedge k_\alpha \rightarrow \left(\mathbf{k} \times \hat{\mathbf{t}}\right) \times \bar{\mathbf{S}}$.[16] Also we let $\omega^\mu{}_\alpha t^\alpha \rightarrow \boldsymbol{\omega} \times \hat{\boldsymbol{t}}$[17]. For simplicity, let us assume that the congruence has vanishing shear and expansion.[18] For this case the two different approaches to spatial curvature radius coincide and we will drop any instances of subscripts 'ns' or 'ps'. Introduce then $\mathbf{a}_{\text{gyro}} = \frac{d^2\mathbf{x}}{d\tau_0^2}$, where $\mathbf{x}$ and $\tau_0$ are the inertial coordinates of a system locally comoving with the congruence[19]. Also denoting the acceleration of the reference congruence relative to an inertial system locally comoving with the reference congruence by $\mathbf{a}_{\text{ref}}$, we get from (45)[20]

$$\frac{D\bar{\boldsymbol{S}}}{D\tau} = \frac{\gamma^3}{\gamma+1}\left(\left[\mathbf{a}_{\text{gyro}} + \frac{1}{\gamma}(\mathbf{a}_{\text{ref}} + \boldsymbol{\omega} \times \mathbf{v})\right] \times \mathbf{v}\right) \times \bar{\mathbf{S}} - \gamma\boldsymbol{\omega} \times \bar{\mathbf{S}}. \tag{46}$$

This is a perfect match with the result of the intuitive derivation performed in [2].

Analogously we may study (44) for the particular case of vanishing shear, thus considering a rigid congruence. The three-dimensional version of this equation then becomes

$$\frac{D\bar{\mathbf{S}}}{D\tau} = \left[\gamma^2 v(\mathbf{a}_{\text{ref}} \times \hat{\mathbf{t}}) - \gamma\boldsymbol{\omega} + (\gamma-1)(2\gamma+1)(\boldsymbol{\omega} \times \hat{\mathbf{t}}) \times \hat{\mathbf{t}}\right.$$
$$\left. + \gamma v(\gamma - 1)\left(\frac{\hat{\mathbf{n}}}{R} \times \hat{\mathbf{t}}\right)\right] \times \bar{\mathbf{S}}. \tag{47}$$

We may simplify this expression a bit by introducing $\boldsymbol{\omega} = \boldsymbol{\omega}_\parallel + \boldsymbol{\omega}_\perp$, where $\parallel$ and $\perp$ means parallel respectively perpendicular to $\hat{\mathbf{t}}$. Also using $\mathbf{v} = v\hat{\mathbf{t}}$ we readily find

$$\frac{D\bar{\mathbf{S}}}{D\tau} = \left[\gamma^2(\mathbf{a}_{\text{ref}} \times \mathbf{v}) - \gamma\left(\boldsymbol{\omega}_\parallel + \left(2\gamma - \frac{1}{\gamma}\right)\boldsymbol{\omega}_\perp\right)\right.$$
$$\left. + \gamma(\gamma-1)\left(\frac{\hat{\mathbf{n}}}{R} \times \mathbf{v}\right)\right] \times \bar{\mathbf{S}}. \tag{48}$$

Again this is a perfect match with the intuitive formalism of [2].

[16] Strictly speaking, what we mean by the cross product $\mathbf{a} \times \mathbf{b}$ of two three-vectors $\mathbf{a}$ and $\mathbf{b}$ is $[\text{Det}(g_{ij})]^{-\frac{1}{2}}\epsilon^{ijk}a_j b_k$ where the indices have been lowered with the local three-metric (again assuming local coordinates orthogonal to the congruence). Notice that in general (for congruences with rotation) there are no global time-slices that are orthogonal to the congruence. The local three-metric corresponding to local orthogonal coordinates is however well defined everywhere anyway. For a shearing (expanding) congruence it will however be time dependent (whatever global time slices we choose).

[17] Letting $\omega^\mu = (0, \boldsymbol{\omega})$ in coordinates locally comoving with the congruence, we have $\omega^\mu = \frac{1}{2}\frac{1}{\sqrt{g}}\eta_\sigma\epsilon^{\sigma\mu\gamma\rho}\omega_{\gamma\rho}$, where $g = -\text{Det}[g_{\alpha\beta}]$ and $\epsilon^{\sigma\mu\gamma\rho}$ is $+1$, $-1$ or 0 for $\sigma\mu\gamma\rho$ being an even, odd or no permutation of $0, 1, 2, 3$ respectively.

[18] This incidentally implies that the 'orthogonal' three-metric mentioned in a previous footnote is time independent.

[19] Working in another set of spatial coordinates $\mathbf{a}_{\text{gyro}}$ naturally transforms as a three-vector.

[20] Notice that $D/D\tau$ corresponds to covariant differentiation with respect to the three-metric.

*8.1. The rotation vector relative to the reference observers.*

Letting $\tau_0 = \gamma d\tau$ denote local time for an observer comoving with the congruence we can write (46) and (48) respectively as

$$\frac{D\bar{\mathbf{S}}}{D\tau_0} = \mathbf{\Omega} \times \bar{\mathbf{S}}. \tag{49}$$

Here $\mathbf{\Omega}$ is given by (46)

$$\mathbf{\Omega} = \frac{\gamma^2}{\gamma+1}(\mathbf{a}_{\text{gyro}} \times \mathbf{v}) + \frac{\gamma}{\gamma+1}(\mathbf{a}_{\text{ref}} \times \mathbf{v}) - \boldsymbol{\omega}_{\parallel} - \left(2 - \frac{1}{\gamma}\right)\boldsymbol{\omega}_{\perp}. \tag{50}$$

This form is practical if the gyroscope is freely falling, in which case $\mathbf{a}_{\text{gyro}} = 0$. Alternatively we can get $\mathbf{\Omega}$ from (48)

$$\mathbf{\Omega} = \gamma(\mathbf{a}_{\text{ref}} \times \mathbf{v}) - \boldsymbol{\omega}_{\parallel} - \left(2\gamma - \frac{1}{\gamma}\right)\boldsymbol{\omega}_{\perp} + (\gamma - 1)\left(\frac{\hat{\mathbf{n}}}{R} \times \mathbf{v}\right). \tag{51}$$

This form is practical if the gyroscope follows some predetermined path while being acted on by forces.

*8.2. A note on the gyroscope axis and the projected spin vector*

From the simple relation (see section 4.8) between the stopped spin vector and the projected spin vector and the gyroscope axis respectively, we can use the law of rotation for the stopped spin vector to derive corresponding differential equations for $\mathbf{S}$ and $\mathbf{X}$

$$\frac{d\mathbf{S}}{d\tau_0} = \gamma^2\mathbf{v}\left[\mathbf{S} \cdot \frac{d\mathbf{v}}{d\tau_0}\right] + \mathbf{\Omega}_{\text{e}\parallel} \times [\mathbf{S}]_{\perp} + \mathbf{\Omega}_{\text{e}\perp} \times \left(\frac{1}{\gamma}[\mathbf{S}]_{\parallel} + \gamma[\mathbf{S}]_{\perp}\right) \tag{52}$$

$$\frac{d\mathbf{X}}{d\tau_0} =- \gamma^2\frac{d\mathbf{v}}{d\tau_0}[\mathbf{X} \cdot \mathbf{v}] + \mathbf{\Omega}_{\text{e}\parallel} \times [\mathbf{X}]_{\perp} + \mathbf{\Omega}_{\text{e}\perp} \times \left(\gamma[\mathbf{X}]_{\parallel} + \frac{1}{\gamma}[\mathbf{X}]_{\perp}\right). \tag{53}$$

Here we have for brevity introduced

$$\mathbf{\Omega}_{\text{e}} = \gamma(\mathbf{a}_{\text{ref}} \times \mathbf{v}) - \boldsymbol{\omega}_{\parallel} - \left(2\gamma - \frac{1}{\gamma}\right)\boldsymbol{\omega}_{\perp}. \tag{54}$$

Note that the $\frac{d\mathbf{v}}{d\tau_0}$ entering (52) and (53) is the velocity derivative relative to the reference frame (not relative to a freely falling frame). We note that these differential equations are more complicated than the ones for the stopped spin vector. We conclude that if we are interested in $\mathbf{S}$ or $\mathbf{X}$, it is likely wise to first solve the equation for the stopped spin vector and then (as in section 4.4) use the result to find $\mathbf{S}$ or $\mathbf{X}$.

## 9. Motion along a straight line in static geometry

As a first example of how one may use the derived formalism, consider a train moving along a straight spatial line in some static geometry. In fact, to be specific, we may consider the train to be moving relative to an upwards accelerating platform in special relativity. On the train we have suspended a gyroscope so that there are no torques acting on it in the comoving system. We thus consider gyroscope motion along a straight line, with respect to a non-rotating but accelerating reference frame. Letting $\mathbf{g} = -\mathbf{a}_{\mathrm{ref}}$ and $\tau_0 = \gamma\tau$, (48) is immediately reduced to

$$\frac{d\bar{\mathbf{S}}}{d\tau_0} = -\gamma\left(\mathbf{g} \times \mathbf{v}\right) \times \bar{\mathbf{S}}. \tag{55}$$

We understand that a gyroscope initially pointing in the forward direction will tip forward as depicted in figure 7.

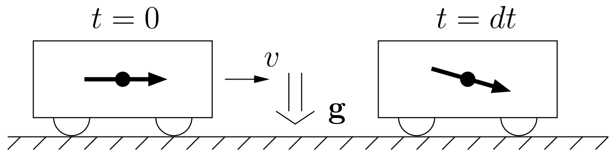


**Figure 7.** A train moving relative to a straight platform with proper upward acceleration. A gyroscope with a torque free suspension will precess clockwise (for positive $v$).

Note that the stopped spin vector with respect to the platform corresponds precisely to the spin vector as perceived relative to the train. For example, if the stopped spin vector points 45° down from the horizontal direction, the gyroscope as seen from the train points 45° down from the horizontal direction. To express the gyroscope precession with respect to coordinates comoving with the train we therefore just let $\tau_0 \rightarrow \gamma\tau$ in (55) and we have the precession explicitly in terms of the time $\tau$ on the train. Relative to the train, the gyrocope thus precesses at a steady rate given by $\Omega_{\text{relative train}} = \gamma^2 v g$. This means that the train in fact has a proper rotation, but more on this is given in [2].

We can parameterize the gyroscope trajectory by the distance $s$ along the platform rather than the time $\tau_0$. Then (55) can be expressed as

$$\frac{d\bar{\mathbf{S}}}{ds} = -\gamma\left(\mathbf{g} \times \hat{\mathbf{t}}\right) \times \bar{\mathbf{S}}. \tag{56}$$

Assuming the train velocity to be low, the tipping angle per distance traveled is thus independent of the velocity. We have simply $d\alpha/ds \simeq g$. Thus on a stretch of length $\delta s$ we get a net rotation $\delta\alpha$

$$\delta\alpha \simeq g\delta s. \tag{57}$$

If we want to express $\delta s$ and $g$ and in SI units we must divide the right hand side by $c^2$ (expressed in SI units). Setting $\delta s = 10^3$ m and $g = 9.81$ m/s$^2$ we get simply

$$\delta\alpha = \frac{9.81 \cdot 10^3}{(3 \cdot 10^8)^2} \approx 1 \cdot 10^{-13}(\text{rad}). \tag{58}$$

This is quite a small angle, and we understand that the relativistic effects of gyroscope precession for most cases here at Earth are small. Notice how simple this calculation was in the stopped spin vector three-formalism.

## 10. Axisymmetric spatial geometries, and effective rotation vectors

The equations (46) and (48) both describe how the gyroscope rotates with respect to a coordinate frame that is parallel transported with respect to the spatial geometry. Suppose then that we consider motion in the equatorial plane of some axisymmetric geometry. As a specific example we may want to know the net rotation of the gyroscope relative to its initial configuration after a full orbit (not necessarily a circular orbit). We must then take into consideration that a parallel transported frame in general will be rotated relative to its initial configuration after a complete orbit.

We can however introduce a new reference frame, that rotates relative to local coordinates spanned by $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$, in the same manner as a parallel transported reference frame does on a plane. In other words, if we for instance consider a counterclockwise displacement $(\delta\varphi,\delta r)$, then relative to the local vectors $\hat{\mathbf{r}}$ and $\delta\hat{\boldsymbol{\varphi}}$, the new reference frame should turn precisely $\delta\varphi$ clockwise. Such a reference frame would always return to its initial configuration after a full orbit.

To find the rotation of the new reference frame with respect to a parallel transported frame, we first investigate how a vector that is parallel transported with respect to the curved axisymmetric geometry rotates relative to the local coordinates spanned by $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$.

### *10.1. The rotation of a parallel transported vector relative to $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$*

The line element for a two-dimensional axisymmetric spatial geometry can be written in the form[21]

$$ds^2 = g_{rr}dr^2 + r^2 d\varphi^2. \tag{59}$$

As depicted in figure 8 we can imagine an embedding of the geometry, where we cut out a small section and put it on a flat plane. What we want is an expression for how much a vector that is parallel transported, for example along the depicted straight dashed line, rotates relative to the local coordinates $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$. We understand that the rotation angle corresponds to the angle $\delta\alpha$ as depicted.
Using the notations introduced in figure 8 we have simply

$$R_0\delta\alpha = r\delta\varphi \tag{60}$$

$$(R_0 + ds)\delta\alpha = (r + dr)\delta\varphi. \tag{61}$$

Eliminating $R_0$ and using $ds = \sqrt{g_{rr}}dr$ it follows readily that

$$\delta\alpha = \frac{\delta\varphi}{\sqrt{g_{rr}}}. \tag{62}$$

[21] Note that if we consider for instance a Kerr black hole, where we (in standard representation) have $d\varphi dt$-terms, we cannot simply select the spatial terms (without $dt$) to get the spatial line element. There is however an effective spatial geometry also for this case. We may derive the form of this geometry by for instance sending photons back and forth between nearby spatial points and checking the proper time that passes.

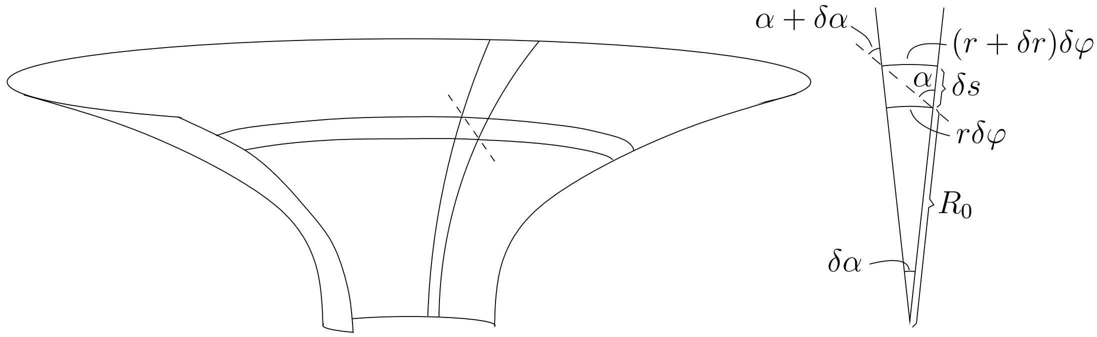


**Figure 8.** Cutting out a section of a certain $d\varphi$ and $dr$ of the embedded geometry (to the left) and putting it on a flat plane (to the right). Note that $r$ is the circumferential radius, and $R_0$ is the radius of curvature for a circle at the $r$ in question (not to be confused with the $R$ of the trajectory along which we are parallel transporting the vector)

So this tells us how a parallel transported vector turns relative to the local $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$, for a small displacement in $\varphi$ and $r$.

*10.2. The new reference frame, and the effective rotation vector*

On a flat plane, the corresponding expression to (62) is of course simply

$$\delta\alpha = \delta\varphi. \tag{63}$$

A reference frame that with respect to $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$ rotates as if we had a flat geometry would then according to (62) and (63) rotate relative to a parallel transported reference frame with an angular frequency (never mind the sign for the moment)

$$\omega_{\text{space}} = \frac{d\varphi}{d\tau_0}\left(\frac{1}{\sqrt{g_{rr}}} - 1\right). \tag{64}$$

Note also that we have

$$\left|\frac{d\varphi}{d\tau_0}\right| = \left|\frac{r d\varphi}{d\tau_0}\right|\frac{1}{r} = \frac{1}{r}|\mathbf{v}\cdot\hat{\boldsymbol{\varphi}}| = \frac{1}{r}|\mathbf{v}\times\hat{\mathbf{r}}|. \tag{65}$$

Thinking about the sign for a second, we realize that with respect to the 'would-be-flat' reference frame, a parallel transported reference frame will have a rotation given by

$$\boldsymbol{\omega}_{\text{space}} = \frac{1}{r}\left(\frac{1}{\sqrt{g_{rr}}} - 1\right)\mathbf{v}\times\hat{\mathbf{r}}. \tag{66}$$

Knowing that infinitesimal rotations can simply be added (to lowest order), using (66) together with (51) and letting $\mathbf{g} = -\mathbf{a}_{\text{ref}}$, we get the gyroscope rotation relative to the 'would-be-flat'-grid as[22]

$$\boldsymbol{\Omega}_{\text{effective}} = (\gamma - 1)\left(\frac{\hat{\mathbf{n}}}{R} \times \mathbf{v}\right) - \gamma(\mathbf{g} \times \mathbf{v}) - \boldsymbol{\omega}_{\parallel} - \boldsymbol{\omega}_{\perp}\left(2\gamma - \frac{1}{\gamma}\right) + \frac{1}{r}\left(\frac{1}{\sqrt{g_{rr}}} - 1\right)\mathbf{v} \times \hat{\mathbf{r}}. \tag{67}$$

We can integrate this equation to find the net precession of a gyroscope transported along any path in the axisymmetric geometry.

### 10.3. Comments on the integrability

As a particular application of (67), we can consider the net precession of a gyroscope transported along some closed orbit. Since the 'would-be-flat' reference frame returns to its original configuration after a full turn, we just integrate the effects of the infinitesimal rotations following from (67) to calculate the net turn. Notice however that to do this straightforwardly, we need $\boldsymbol{\Omega}_{\text{effective}}$ in the coordinate base of the reference frame (i.e. the would-be-flat frame). In general we however only have $\boldsymbol{\Omega}_{\text{effective}}$ in the coordinates adapted to the stationary observers. For most cases where we would be interested in motion in an axisymmetric geometry, like motion in the equatorial plane of a Kerr black hole, this however presents no problem. Then all rotations are in the plane of motion and the rotation vector $\boldsymbol{\Omega}_{\text{effective}}$ is constant (in the $\hat{z}$-direction) in the coordinate basis of the reference frame. Notice that the $\tau_0$ implicitly entering these equations in the $\boldsymbol{\Omega}_{\text{effective}}$ is the proper time for a stationary observer. If we instead want to express the precession in global (Schwarzschild) time, we just multiply by the time dilation factor.

Even assuming all rotations to be in the plane of motion, we must still in general integrate to get the net precession of the gyroscope[23]. For the particular case of circular motion with constant speed, assuming the time dilation (i.e. the lapse), $\boldsymbol{\omega}$ and $\mathbf{g} \cdot \hat{\mathbf{r}}$ to be independent of $\varphi$ (as is the case for the equatorial plane of a Kerr black hole), there is however no need to integrate at all since all the terms of (67) are constant. The result follows immediately, assuming that we know $\boldsymbol{\omega}$, $\boldsymbol{g}$ and $g_{rr}$. Incidentally it follows from (60) and (61) that $R = r\sqrt{g_{rr}}$ for circular motion.

### 10.4. Comment regarding $\boldsymbol{g}$, $\boldsymbol{\omega}$ and $g_{rr}$

The reference background (fixed to the stars) around a spinning planet, like the Earth, is both accelerating and curved. Also there is frame dragging due to the planet rotation,

[22] If the geometry in question contains regions where the circumferential radius has a minimum (in 2 dimensions one may call these regions *necks* from the appearance of an embedding of such regions), one can modify (67) a little by introducing a $\pm$ sign in the $\frac{1}{\sqrt{g_{rr}}}$-term. If $\hat{\mathbf{r}}$, which by definition is taken to point away from the center of symmetry, points in the direction of increasing $r$, we choose the positive sign, otherwise the negative sign should be chosen. Note that $\frac{1}{\sqrt{g_{rr}}}$ is zero for radii where the sign changes, so there is no discontinuity in $\boldsymbol{\Omega}_{\text{effective}}$.

[23] Parameterizing the trajectory by some parameter $\lambda$, we understand that time dilation, $R$, $\boldsymbol{\omega}$, $v$ and $\mathbf{g}$ all depends on $\lambda$. Assuming all rotations to be in the plane of motion it is effectively a single (scalar) integral (of the net rotation angle around the $z$-axis).

giving a non-zero rotation of the stationary reference observers. If we have the spacetime metric, we can easily find $\boldsymbol{\omega}$, $\boldsymbol{g}$ and $g_{rr}$. If we do not have an exact spacetime metric however, as is the case for a spinning planet, we need some approximate method (like the Post-Newtonian approximation) to estimate $\boldsymbol{\omega}$, $\boldsymbol{g}$ and $g_{rr}$. Once this is done, assuming the approximation to be valid, (67) gives an accurate description of the precession even considering relativistic speeds.

In the case of a rotating (Kerr) black hole, we do know the metric, and the precession relative to the stationary observers can readily be calculated. Notice that within the ergosphere , there are no stationary (timelike) observers. Still we can in principle use the formalism of this paper also within the ergosphere. To do this we consider coordinates that rigidly rotate around the black hole sufficiently fast to be timelike in the region in question. Indeed for the particular case of circular motion there is a paper [6], that uses this technique.

### *10.5. Free orbit at large radii from a Schwarzschild black hole*

As a simple example, consider a freely falling gyroscope ($\mathbf{a}_{\text{gyro}} = 0$) orbiting in the equatorial plane of a Schwarzschild black hole. Using the static observers as our reference congruence, we have $\boldsymbol{\omega} = 0$. Then it follows from (50) that we have

$$\boldsymbol{\Omega}_{\text{effective}} = \frac{\gamma}{\gamma+1}\mathbf{a}_{\text{ref}} \times \mathbf{v} + \boldsymbol{\omega}_{\text{space}}. \tag{68}$$

The Schwarzschild line element in the equatorial plane is given by

$$d\tau^2 = (1+2\phi)\, dt^2 - (1+2\phi)^{-1}\, dr^2 - r^2 d\varphi^2. \tag{69}$$

Here $\phi = -\frac{M}{r}$. For convenience we now consider large $r$, so that $M/r$ is small. We have then the acceleration of the freely falling frames $g \simeq \frac{\partial \phi}{\partial r}$ (counted positive in the inwards direction). It follows readily, using (66), that for this case we have

$$\begin{aligned}
\boldsymbol{\omega}_{\text{space}} &= \frac{1}{r}\left(\sqrt{1+2\phi}-1\right)\mathbf{v}\times\hat{\mathbf{r}} && (70)\\
&\simeq \frac{\phi}{r}\cdot\mathbf{v}\times\hat{\mathbf{r}} && (71)\\
&\simeq -g\cdot\mathbf{v}\times\hat{\mathbf{r}}. && (72)
\end{aligned}$$

For the large $r$ in question the velocities are low and we may set $\gamma \simeq 1$. Using (72) together with $\mathbf{a}_{\text{ref}} = -\mathbf{g}$ and $\mathbf{g} = -g\hat{\mathbf{r}}$ in (68) gives

$$\begin{aligned}
\boldsymbol{\Omega}_{\text{effective}} &\simeq -\frac{1}{2}(\mathbf{g}\times\mathbf{v}) - \mathbf{g}\times\mathbf{v} && (73)\\
&= -\frac{3}{2}(\mathbf{g}\times\mathbf{v}). && (74)
\end{aligned}$$

This result was derived by W. de Sitter in 1916 (although in a quite different manner than that described here, see [1] p. 1119). We may note that one third of the net effect comes from the spatial geometry. Using a little bit of Newtonian mechanics it is easy to derive that for a satellite orbiting the Earth at a radius $R \simeq R_{\text{Earth}}$, inserting the proper factor of $c$ to handle SI-units, (74) amounts to

$$\begin{aligned}\Omega_{\text{effective}} &= \frac{3}{2c^2}\frac{GM}{R^2}\sqrt{\frac{GM}{R}} \simeq 1.3\cdot 10^{-12}\text{rad/s} \\ &\simeq 4.0\cdot 10^{-5}\frac{\text{rad}}{\text{year}} \simeq 8.3\frac{\text{arcsec}}{\text{year}}.\end{aligned} \tag{75}$$

Knowing that the exterior metric of the Earth is approximately Schwarzschild, we have then an approximation of the effective rotation vector for a gyroscope orbiting the Earth. We can refine this approximation by considering an appropriate non-zero $\boldsymbol{\omega}$, as discussed earlier. Note that, as discussed in section 4.10, the derived precession is the precession with respect to a star-calibrated reference system on the satellite.

In [1] p. 1117-1120, a similar explicitly three-dimensional formalism of spin precession is derived. It is only valid in the Post-Newtonian regime however. The precession given by (67) is however exact (assuming an ideal gyroscope). For instance, considering the above example of free circular motion in a static geometry, we can easily calculate the exact expressions for $g$ and $v$, and thus express the gyroscope precession rate arbitrarily close to the horizon.

## 11. Summary and conclusion

We have seen how we in a covariant manner can derive an effectively three-dimensional spin precession formalism in a general spacetime. In particular the simple form of (16) seems novel.

In [1] p. 1117 a similar approach is taken where they consider only the standard spin vector, but expressed relative to a *boosted* set of base vectors. They however only apply it to the post-Newtonian regime.

As mentioned earlier, Jantzen et. al. ([4, 7, 8]) have already pursued the same general idea, although the specific approach and final form of the results differ. In particular they have not employed the explicit 3-dimensional formalism.

While the general formalism is derived assuming a general congruence, it seems to have its greatest use as a simple three-dimensional formalism assuming a non-shearing congruence. Then we have a fixed spatial geometry and the spatial parallel transport is unambiguous. For this particular case, the derived three-dimensional formalism verifies the result of the intuitive derivation of [2]. We have also given examples of how the three dimensional formalism can be used to easily find results of physical interest.

## Appendix A. Simplifying (14)

In the expansion of the second term within the brackets of (14) there will according to (13) be terms of the type $\frac{Dt^\mu}{D\tau}$. These can be rewritten in terms of $\frac{Du^\mu}{D\tau}$ and $\frac{D\eta^\mu}{D\tau}$ since we have

$$u^\mu = \gamma(\eta^\mu + vt^\mu) \qquad \rightarrow \qquad t^\mu = \frac{u^\mu}{\gamma v} - \frac{\eta^\mu}{v}. \tag{A.1}$$

Dealing with $\frac{Du^\mu}{D\tau}$ rather than $\frac{Dt^\mu}{D\tau}$ is convenient since the former readily can be expressed in terms of spatial curvature and velocity changes relative to the congruence, see [3]. Also $\frac{Du^\mu}{D\tau}$ has a direct physical relevance. Using the identity $\frac{\gamma-1}{\gamma v}=\frac{\gamma v}{\gamma+1}$, it is then easy to derive an alternative form of ${K^\mu}_\alpha$

$$ {K^\mu}_\alpha = {\delta^\mu}_\alpha + \frac{1}{\gamma+1}\left(u_\alpha - \gamma\eta_\alpha\right)\left(\eta^\mu + u^\mu\right). \tag{A.2} $$

Using this in the second term within the brackets of (14) we have

$$ \frac{D{K^\mu}_\alpha}{D\tau} = \frac{D}{D\tau}\left[\frac{1}{\gamma+1}\underbrace{\left(u_\alpha - \gamma\eta_\alpha\right)}_{\gamma v t_\alpha}\underbrace{\left(\eta^\mu + u^\mu\right)}_{(\gamma+1)\eta^\mu+\gamma v t^\mu}\right]. \tag{A.3} $$

As we expand this expression there will be terms containing $\eta^\mu$, $\eta_\alpha$ and $t^\mu t_\alpha$. These we will disregard for the following reasons. Terms containing $\eta_\alpha$ will anyway die when multiplied by $\bar{S}^\alpha$ (as they are in (14)). Terms containing $\eta^\mu$ we will disregard since we for the moment only are interested in ${P^\mu}_\alpha\bar{S}^\alpha = ({g^\mu}_\alpha + \eta^\mu\eta_\alpha)\bar{S}^\alpha$. When we have a neat expression for this we can find the $\eta^\mu$-part a posteriori using the orthogonality of $\bar{S}^\alpha$ and $\eta_\alpha$. We will disregard terms of the type $t^\mu t_\alpha$ since we know that these must cancel anyway for $\bar{S}^\alpha$ to stay normalized (as we know it must by construction of the Fermi transport and the relation to the stopped spin vector)[24]. Note however that in principle, we should contract with the inverse ${K^{-1\nu}}_\mu$[25], before disregarding the terms of the described types (see (14)). The form of the inverse is however such that we can carry out the effective cancellations prior to applying the inverse[26]. We then readily find

$$ \frac{D{K^\mu}_\alpha}{D\tau} \overset{\text{eff}}{=} \frac{\gamma v}{\gamma+1}\left(\left[\frac{Du_\alpha}{D\tau} - \gamma\frac{D\eta_\alpha}{D\tau}\right]_\perp t^\mu + t_\alpha\left[\frac{Du^\mu}{D\tau} + \frac{D\eta^\mu}{D\tau}\right]_\perp\right). \tag{A.4} $$

By the perpendicular sign $\perp$ we here mean that we should select only the part orthogonal to both $t^\mu$ and $\eta^\mu$. By $\overset{\text{eff}}{=}$ we indicate that the equality holds excepting terms of the type $\eta^\mu$, $\eta_\alpha$ and $t^\mu t_\alpha$. In an analogous manner we readily find for the first term within

[24] From normalization follows that $\bar{S}_\alpha\frac{D\bar{S}^\alpha}{D\tau}=0$. For the particular case where $\bar{S}^\alpha = \bar{S}t^\alpha$ momentarily, it follows that any net term of the form $at^\mu t_\alpha$ in the right hand side of (14) must vanish. Since the parameter $a$ does not depend on $S^\alpha$ it follows that it must vanish entirely. The point is that the form of (A.3) is such that, when expanded it can be written as a sum of tensors of the type $A^\mu B_\alpha$. Letting the suffix $\perp$ indicate that only the part orthogonal to both $\eta^\mu$ and $t^\mu$ should be selected, each such term can be written in the form $(t^\mu t_\rho A^\rho + [A^\mu]_\perp)(t_\alpha t^\sigma B_\sigma + [B_\alpha]_\perp)$. Adding up the resulting terms of the type $t^\mu t_\alpha$ (including the terms of this type coming from the first term within the brackets of (14)) into a single term $at^\mu t_\alpha$ we *know* that $a$ must be zero.

[25] Note from (13) that the effect of contracting ${K^{-1\mu}}_\alpha$ with a contravariant vector is that it shortens the $t^\mu$-component of the vector by a $\gamma$-factor, while the rest of the on-slice (orthogonal to $\eta^\mu$) part of thee vector is unaffected.

[26] If the inverse had contained for instance terms of the type $t^\nu\eta_\mu$ – we could not cancel $\eta^\mu$ terms directly within the brackets of (14). That the inverse is not containing any such terms is a benefit of the particular gauge choice in choosing ${K^\nu}_\mu$ – where we had a freedom to include any terms containing $\eta_\mu$.

brackets of (14)

$$u^\mu K^\rho{}_\alpha \frac{D u_\rho}{D\tau} \overset{\text{eff}}{=} \gamma v t^\mu \left[\frac{D u_\alpha}{D\tau}\right]_\perp . \tag{A.5}$$

Now use (A.4) and (A.5) in (14). Shorten the $t^\mu$ components by a $\gamma$ factor (according to the effect of the inverse), and neglect the $\eta^\mu$-term. We readily find

$$P^\mu{}_\alpha \frac{D\bar{S}^\alpha}{D\tau} = \frac{\gamma v}{\gamma+1}\bar{S}^\alpha \left( t^\mu \left[\frac{D}{D\tau}(u_\alpha + \eta_\alpha)\right]_\perp - t_\alpha \left[\frac{D}{D\tau}(u^\mu + \eta^\mu)\right]_\perp \right). \tag{A.6}$$

Now that we have this compact expression we may also find the $\eta^\mu$ term that we earlier omitted. From orthogonality, $\bar{S}^\alpha \eta_\alpha = 0$, follows that $\frac{D\bar{S}^\alpha}{D\tau}\eta_\alpha = -\bar{S}^\alpha \frac{D\eta_\alpha}{D\tau}$ which gives

$$\begin{aligned} \frac{D\bar{S}^\mu}{D\tau} = \frac{\gamma v}{\gamma+1}\bar{S}^\alpha \left( t^\mu \left[\frac{D}{D\tau}(u_\alpha + \eta_\alpha)\right]_\perp - t_\alpha \left[\frac{D}{D\tau}(u^\mu + \eta^\mu)\right]_\perp \right) \\ + \eta^\mu \bar{S}^\alpha \frac{D\eta_\alpha}{D\tau}. \end{aligned} \tag{A.7}$$

So here we have the transport equation for the stopped spin vector (giving the rotation relative to inertial coordinates).

## Appendix B. A note concerning the intrinsic angular momentum

As an idealized scenario we consider a special relativistic gyroscope which we model as an isolated system of point particles undergoing four-momentum conserving internal collisions. Following the discussion in [9] p. 87-90, we define the angular momentum tensor with respect to the spacetime origin as

$$L^{\mu\nu} = \sum x^\mu p^\nu - x^\nu p^\mu. \tag{B.1}$$

Here the summation runs over events $x^\mu$ and four-momenta $p^\mu$ for the various particles at a particular time slice $t = const$. The (Pauli-Lubanski) spin vector can be written as

$$S_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} L^{\nu\rho} V^\sigma \tag{B.2}$$

Here $V^\mu$ is the four-velocity of the center of mass and $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor (density) where $\epsilon_{xyz0} = 1$. Furthermore we introduce an angular momentum four-vector $h^\mu := (0, \mathbf{h})$, where $\mathbf{h}$ is the standard (relativistic) angular momentum three-vector, with respect to our reference coordinates. Defining $\eta^\mu$ as a purely time directed normalized vector with respect to the reference coordinates, we can write

$$h_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} L^{\nu\rho} \eta^\sigma. \tag{B.3}$$

Letting $\mathbf{v}$ be the velocity of the center of mass, $\gamma$ the corresponding gamma factor and setting $(0, \mathbf{v}) := v t^\mu$ with respect to the reference coordinates, we can decompose the four-velocity of the center of mass as $V^\mu = \gamma(\eta^\mu + v t^\mu)$. Using this in (B.2) together with (B.3), it follows that

$$S_\mu = \gamma h_\mu + \gamma v \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} L^{\nu\rho} t^\sigma. \tag{B.4}$$

It is a short exercise to show that in three-formalism this amounts to

$$\mathbf{h} = \mathbf{S}/\gamma + \mathbf{r}_{\mathrm{c}} \times \mathbf{p}. \tag{B.5}$$

Here $\mathbf{h}$ is the net angular momentum of the system of point particles, $\mathbf{r}_{\mathrm{c}}$ is the center of mass (center of energy), $\gamma$ is the gamma factor for the velocity of the center of mass, $\mathbf{p}$ is the net relativistic three-momentum and $\mathbf{S}$ is the spatial part of the spin vector. Note that the intrinsic angular momentum is not given by $\mathbf{S}$ but by $\mathbf{S}/\gamma$. Note incidentally also that there is a difference between the center of mass and the *proper* center of mass (see [9] p. 87-90). As pointed out e.g. in [10], the gyroscope center of mass does not in general lie on the gyroscope central axis.

A real gyroscope moving under the influence of forces is neither (simply) consisting of point particles nor is it isolated. A more detailed analysis would likely assume a general energy momentum tensor $T^{\mu\nu}$ and allow for external forces acting on the elements of the gyroscope. For the purposes of this article the simple derivation outlined above will however suffice.

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