

Generalizing Optical Geometry

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Abstract. We show that by employing the standard projected curvature as a measure of spatial curvature, we can make a certain generalization of optical geometry (Abramowicz and Lasota 1997 *Class. Quantum Grav.* **14** A23). This generalization applies to any spacetime that admits a hypersurface orthogonal shearfree congruence of worldlines. This is a somewhat larger class of spacetimes than the conformally static spacetimes assumed in standard optical geometry. In the generalized optical geometry, which in the generic case is time dependent, photons move with unit speed along spatial geodesics and the sideways force experienced by a particle following a spatially straight line is independent of the velocity. Also gyroscopes moving along spatial geodesics do not precess (relative to the forward direction). Gyroscopes that follow a curved spatial trajectory precess according to a very simple law of three-rotation. We also present an inertial force formalism in coordinate representation for this generalization. Furthermore, we show that by employing a new sense of spatial curvature (Jonsson 2006 *Class. Quantum Grav.* **23** 1) closely connected to Fermat's principle, we can make a more extensive generalization of optical geometry that applies to arbitrary spacetimes. In general this optical geometry will be time dependent, but still geodesic photons move with unit speed and follow lines that are spatially straight in the new sense. Also, the sideways experienced (comoving) force on a test particle following a line that is straight in the new sense will be independent of the velocity.

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1. Introduction

General Relativity is a theory about curved spacetime. Nevertheless we can often gain insight by splitting spacetime into space and time [1]. To do this we may introduce a foliation of spacetime into spatial hypersurfaces, henceforth referred to as time slices, and study the geometry on these slices. In general this spatial geometry will be time dependent, where time is a parameter that we associate with the different slices.

For the particular case of conformally static spacetimes, Abramowicz et. al. [2]-[7], have demonstrated that it can be fruitful to study a certain rescaled version of the spatial geometry, known as the optical geometry. This rescaled geometry (which is static) has several features that in general the non-rescaled spatial geometry lacks:

- A photon moves with unit speed (with respect to the preferred time coordinate).
- A photon corresponding to a spacetime null geodesic follows a spatial geodesic.
- An observer following a spatial geodesic will experience an acceleration (force), perpendicular to the direction of motion, that is independent of the velocity.
- A gyroscope following a spatial geodesic will not precess relative to the forward direction of motion.

The optical geometry allows us to explain, in a pictorial manner, several interesting features of black holes [1]. For example a gyroscope orbiting close to a black hole will precess (relative to the forward direction of motion) in the opposite direction from that of a gyroscope in orbit far away from the hole. Another feature is that a rocket orbiting the black hole on a circle near the horizon will require a higher outwards directed rocket thrust the faster it orbits the hole, contrary to the situation far from the hole.

The optical geometry, with the above features, has however only been successfully constructed in conformally static spacetimes. The question then arises if one can generalize it to incorporate a larger class of spacetimes. We will show that this is indeed possible.

This article rests in part on the results of papers [8] and [9]. Where appropriate we will briefly review the necessary formalism of those papers.

1.1. The basic notation

In a general spacetime, consider a reference congruence of timelike worldlines of four-velocity η^μ . We can split the four-velocity v^μ of a test particle into a part parallel to η^μ and a part orthogonal to η^μ

$$v^\mu = \gamma(\eta^\mu + vt^\mu). \quad (1)$$

Here v is the speed of the test particle relative to the congruence and γ is the corresponding γ -factor. The vector t^μ is a normalized spatial vector (orthogonal to η^μ), pointing in the (spatial) direction of motion.

We will also use the kinematical invariants of the congruence defined for a timelike vector field η^μ as [10]³

$$a_\mu = \eta^\alpha \nabla_\alpha \eta_\mu \quad (2)$$

$$\theta = \nabla_\alpha \eta^\alpha \quad (3)$$

³As concerns $\omega_{\mu\nu}$ and $\sigma_{\mu\nu}$, the projection operators in the definition given in [10] enters in a slightly different way than as presented here. For normalized vector fields η^μ the definitions are equivalent, but the form presented here is more useful if the vector field in question is not normalized. Note also that the sign on $\omega_{\mu\nu}$ is a matter of convention.

$$\sigma_{\mu\nu} = \frac{1}{2}P^\rho{}_\nu P^\kappa{}_\mu (\nabla_\rho \eta_\kappa + \nabla_\kappa \eta_\rho) - \frac{1}{3}\theta P_{\mu\nu} \quad (4)$$

$$\omega_{\mu\nu} = \frac{1}{2}P^\rho{}_\nu P^\kappa{}_\mu (\nabla_\rho \eta_\kappa - \nabla_\kappa \eta_\rho). \quad (5)$$

In order of appearance these objects denote the acceleration vector, the expansion scalar, the shear tensor and the rotation tensor. We will also employ what we may denote the expansion-shear tensor

$$\theta_{\mu\nu} = \frac{1}{2}P^\rho{}_\nu P^\kappa{}_\mu (\nabla_\rho \eta_\kappa + \nabla_\kappa \eta_\rho). \quad (6)$$

Throughout the article we will use $c = 1$ and adopt the spatial sign convention $(-, +, +, +)$. The projection operator⁴ along the congruence then takes the form $P^\alpha{}_\beta \equiv \delta^\alpha{}_\beta + \eta^\alpha \eta_\beta$. Vectors that are orthogonal to η^μ , will be referred to as *spatial* vectors. We will also find it convenient to introduce the suffix \perp . When applied to a four-vector, like $[K^\mu]_\perp$, it selects the part within the brackets that is perpendicular to both η^μ and t^μ ⁵.

2. Generalizing the optical geometry

A key feature of optical geometry is that we have a *space* that accounts for the motion of geodesic photons. To define a space, given a spacetime, we specify a congruence of timelike worldlines (generated by a normalized vector field η^μ). Every worldline corresponds to a single point in the space.

To completely account for the behavior of photons we need also to introduce a (global) time in which the position of photons in the spatial geometry is evolved. This is done by introducing time slices. These time slices can be defined by a single function $t(x^\mu)$ (where the slices are defined by $t = \text{const}$)⁶.

It is easy to realize that if we want the velocity⁷ of photons to be independent of direction we must have the time slices orthogonal to the congruence. This means that, given $t(x^\mu)$, the local direction of the congruence is uniquely determined by

$$\eta_\mu = -e^\Phi \nabla_\mu t. \quad (7)$$

The function Φ can be determined by demanding $\eta^\mu \eta_\mu = -1$. Then we can (in principle) integrate (7) to find the congruence lines uniquely.

Photon geodesics are invariant under conformal rescalings of the metric. Thus without affecting the spacetime properties with respect to geodesic photons, we can

⁴Applying this operator to a vector extracts the part of the vector that is orthogonal to η^μ .

⁵Hence $[K^\mu]_\perp = P^\mu{}_\alpha (K^\alpha - K^\beta t_\beta t^\alpha)$.

⁶We can introduce spacelike time slices within a finite region around any point in an arbitrary spacetime. In globally hyperbolic spacetimes such slices can be globally defined.

⁷We have not introduced a three-metric yet so we cannot really talk about velocities. Still given two nearby congruence lines A and B, we can compare the coordinate time (t) it takes a photon to move from A to B and vice versa. If the time slice is not orthogonal to the congruence then t_{AB} will not in general equal t_{BA} for infinitesimally displaced congruence lines. Thus the velocity would be dependent on the spatial direction of motion.

rescale the metric around every spacetime point with a factor $e^{-2\Phi}$. Letting a tilde denote objects related to the rescaled spacetime, we have $\tilde{g}_{\mu\nu} = e^{-2\Phi} g_{\mu\nu}$. The conformal rescaling effectively removes time dilation (lapse) so that $dt = d\tilde{\tau}_0$, where $d\tilde{\tau}_0$ is the proper time along the congruence in the rescaled spacetime. Relative to the rescaled space, photons move with unit speed $\frac{ds}{dt} = 1$. Thus the first point in the list of features in the introduction (photons move with constant speed with respect to coordinate time), we can always achieve. What about the second point? Under what conditions will photons follow straight spatial lines, and indeed what do we mean by following a straight line if the spatial geometry is time dependent?

3. Generalizing the optical geometry using the projected curvature

As regards what is spatially straight, most likely the first thing that comes to mind (at least it was for the authors of this article), is to consider a projection of the null trajectory in question down along the congruence to the local slice⁸. If the spatial curvature of the projected curve vanishes – then we say that the trajectory is straight. In [8] a general formalism of inertial forces in terms of the projected curvature is derived using an arbitrary congruence of timelike worldlines. The (projected) four-acceleration of the test particle in question can be decomposed as (see section 1.1 concerning notation)

$$\frac{1}{\gamma^2} P^\mu{}_\alpha \frac{Dv^\alpha}{D\tau} = a^\mu + 2v [t^\alpha \nabla_\alpha \eta^\mu]_\perp + vt^\mu t^\alpha t^\rho \nabla_\rho \eta_\alpha + \gamma \frac{dv}{d\tau} t^\mu + v^2 \frac{n^\mu}{R}. \quad (8)$$

Here R is the projected spatial curvature radius and n^μ is a normalized spatial vector (orthogonal to both t^μ and η^μ), pointing in the direction of projected spatial curvature (the principal normal). The left hand side of (8) can be expressed in terms of the forces acting on the test particle. We have [8]

$$P^\mu{}_\alpha \frac{Dv^\alpha}{D\tau} = \frac{1}{m} P^\mu{}_\alpha f^\alpha = \frac{1}{m} (\gamma F_\parallel t^\mu + F_\perp m^\mu). \quad (9)$$

Here m^μ is a normalized spatial vector orthogonal to both t^μ and η^μ . F_\perp and F_\parallel are respectively the forces perpendicular and parallel to the direction of motion, as experienced in a system comoving with the test particle in question. The right hand side of (8) can be expressed in terms of the kinematical invariants of the congruence, through the identity [10] $\nabla_\nu \eta_\mu = \omega_{\mu\nu} + \theta_{\mu\nu} - a_\mu \eta_\nu$. Then (8) takes the form [8]

$$\begin{aligned} \frac{1}{m\gamma^2} (\gamma F_\parallel t^\mu + F_\perp m^\mu) &= a^\mu + 2v [t^\beta (\omega^\mu{}_\beta + \theta^\mu{}_\beta)]_\perp + vt^\alpha t^\beta \theta_{\alpha\beta} t^\mu \\ &\quad + \gamma \frac{dv}{d\tau} t^\mu + v^2 \frac{n^\mu}{R}. \end{aligned} \quad (10)$$

On the right hand side we have first three terms that enter as inertial forces (if we multiply them by $-m$), and the last two terms describe the motion (acceleration) relative to the reference congruence.

⁸If the congruence has no rotation there exists a finite sized slicing orthogonal to the congruence. If the congruence is rotating we can still introduce a slicing that is orthogonal at the point in question. It is easy to realize that whatever such locally orthogonal slicing we choose, the projected curvature and curvature directions will be the same, and are thus well defined.

Following [8] one can form a corresponding rescaled version of (8) by putting a tilde on everything in (8). Next one finds the general relation between rescaled and non-rescaled four-acceleration, the result is given by given by (A.2). Setting $\tilde{a}^\mu = 0$ and $\tilde{\omega}_{\mu\nu} = 0$ as is appropriate for the congruence and rescaling at hand, also using $\tilde{t}^\mu = e^\Phi t^\mu$ and $\tilde{m}^\mu = e^\Phi m^\mu$ and using (9) one readily gets [8]

$$\begin{aligned} \frac{1}{m\gamma^2} e^\Phi \left(\frac{F_{\parallel}}{\gamma} \tilde{t}^\mu + F_{\perp} \tilde{m}^\mu \right) &= \frac{1}{\gamma^2} \tilde{P}^{\mu\rho} \tilde{\nabla}_{\rho} \Phi + \frac{v}{\gamma^2} (\tilde{\eta}^{\rho} \tilde{\nabla}_{\rho} \Phi + \tilde{t}^{\alpha} \tilde{t}^{\beta} \tilde{\theta}_{\alpha\beta}) \tilde{t}^\mu \\ &+ 2v \left[\tilde{t}^{\beta} \tilde{\theta}^{\mu}_{\beta} \right]_{\perp} + \frac{dv}{d\tilde{\tau}_0} \tilde{t}^\mu + v^2 \frac{\tilde{n}^\mu}{\tilde{R}}. \end{aligned} \quad (11)$$

Recall that a tilde implies that the object is related to the rescaled spacetime. As concerns γ and v we have however omitted the tilde since these are the same as their non-rescaled analogues. Note also that F_{\parallel} and F_{\perp} are the real (non-rescaled) comoving forces. For a discussion of how to interpret this expression in terms of inertial forces we refer to [8], and the discussion in section 6.1 of this paper.

One sees from (11) (set the left hand side to zero and $v = 1$) that the projected optical curvature of a geodesic photon vanishes, for all spatial directions, if and only if $[\tilde{t}^{\beta} \tilde{\theta}^{\mu}_{\beta}]_{\perp} = 0$ for all \tilde{t}^μ . We also readily see that the sideways (perpendicular) experienced force⁹ F_{\perp} is independent of the velocity when following a trajectory whose projected curvature vanishes, if and only if $[\tilde{t}^{\beta} \tilde{\theta}^{\mu}_{\beta}]_{\perp} = 0$ ¹⁰. As discussed in [8], and reviewed in Appendix B, this holds for all directions \tilde{t}^μ if and only if the congruence is *shearfree*¹¹. Note that for the standard static optical geometry, we have $\tilde{\theta}_{\mu\nu} = 0$, thus the congruence is trivially shearfree. Incidentally, for this case (11) can be written as

$$\frac{e^\Phi}{m} \left(\frac{F_{\parallel}}{\gamma} \tilde{t}^\mu + F_{\perp} \tilde{m}^\mu \right) = \tilde{P}^{\mu\rho} \tilde{\nabla}_{\rho} \Phi + v (\tilde{\eta}^{\rho} \tilde{\nabla}_{\rho} \Phi) \tilde{t}^\mu + \gamma^2 \frac{dv}{d\tilde{\tau}_0} \tilde{t}^\mu + \gamma^2 v^2 \frac{\tilde{n}^\mu}{\tilde{R}}. \quad (12)$$

We conclude that for shearfree congruences we can always manage the first three points of the list of optical geometry features given in the introduction. Now what what about the fourth point, concerning gyroscope precession?

3.1. Gyroscope precession

The spin vector S^μ of an ideal gyroscope transported without any torque acting on it in a comoving system, along a trajectory of four-velocity v^μ , obeys the Fermi-Walker equation

$$\frac{DS^\mu}{D\tau} = v^\mu S_\alpha \frac{Dv^\alpha}{D\tau}. \quad (13)$$

⁹Notice that the force that we are referring to here is the force as received in a system comoving with the test-particle. If we on the other hand consider the reference congruence observers to be providing the sideways force, this force is in fact smaller than the received force by a γ -factor and is hence not independent of the velocity, see section 7.

¹⁰For this case the sideways force is given by $F_{\perp} = me^{-\Phi} \sqrt{\tilde{g}_{\mu\nu} [\tilde{P}^{\mu\rho} \tilde{\nabla}_{\rho} \Phi]_{\perp} [\tilde{P}^{\nu\beta} \tilde{\nabla}_{\beta} \Phi]_{\perp}}$.

¹¹The shear tensor in the rescaled spacetime is given by $\tilde{\sigma}_{\mu\nu} = e^{-\Phi} \sigma_{\mu\nu}$, so the shear tensor in the rescaled spacetime vanishes if and only if it does so in the non-rescaled spacetime.

Introducing $\tilde{S}^\mu = e^\Phi S^\mu$ and $\tilde{v}^\mu = e^\Phi v^\mu$, using the orthogonality of S^μ and v^μ , it is a quick exercise (carried out in Appendix A) to show that this implies

$$\frac{\tilde{D}\tilde{S}^\mu}{\tilde{D}\tilde{\tau}} = \tilde{v}^\mu \tilde{S}_\alpha \frac{\tilde{D}\tilde{v}^\alpha}{\tilde{D}\tilde{\tau}}. \quad (14)$$

Thus a spin vector which is Fermi-Walker transported relative to the non-rescaled spacetime is Fermi-Walker transported also relative to the rescaled spacetime if we just rescale the spin vector itself. In particular, a gyroscope initially pointing in the forward direction t^μ precesses relative to the forward direction in the standard spacetime if and only if it does so in the rescaled spacetime.

In [9], a general formalism of gyroscope precession with respect to a reference congruence is discussed. Here one considers the spin vector that one *would get* if one were to momentarily stop (by a pure boost) the gyroscope with respect to the reference congruence. This spin vector is called the stopped spin vector, denoted by \bar{S}^μ and related to S^μ through

$$\bar{S}^\mu = \left[\delta^\mu{}_\alpha + \eta^\mu \eta_\alpha + \left(\frac{1}{\gamma} - 1 \right) t^\mu t_\alpha \right] S^\alpha. \quad (15)$$

While \bar{S}^μ is orthogonal to the congruence it is not simply the projected part of the standard spin vector (in general both the norm and the spatial direction of these two objects differ). The reason for using \bar{S}^μ rather than S^μ is that \bar{S}^μ (unlike the projected part of S^μ) obeys a simple law of (three-dimensional) rotation

$$\frac{D\bar{S}^\mu}{D\tau} = \frac{\gamma v}{\gamma + 1} \bar{S}^\alpha \left(t^\mu \wedge \left[\frac{D}{D\tau} (v_\alpha + \eta_\alpha) \right]_\perp \right) + \eta^\mu \bar{S}^\alpha \frac{D\eta_\alpha}{D\tau}. \quad (16)$$

Here the last term insures that orthogonality to the congruence is preserved. Note also that $\frac{D}{D\tau}$ means covariant differentiation along the gyroscope worldline. The contraction with the wedge product (defined as $k^\mu \wedge b_\alpha \equiv k^\mu b_\alpha - k^\alpha b_\mu$) corresponds to the three-rotation¹². Furthermore one considers a spacetime analogue of standard spatial transport

$$\frac{Dk^\mu}{D\tau} = \gamma k^\alpha \omega^\mu{}_\alpha + \gamma k^\alpha (\theta^\mu{}_\beta t^\beta \wedge t_\alpha) + \eta^\mu k^\alpha \frac{D\eta_\alpha}{D\tau}. \quad (17)$$

The transport as defined here is norm preserving, and also preserves angles between transported vectors¹³. Considering a spatially straight line ($1/R = 0$), and a vector momentarily aligned with the forward direction, the transport is defined such that the vector remains in the forward direction (analogous to standard spatial transport).

Given (16) and (17) one can form an equation for how fast the stopped spin vector deviates from a corresponding (spatially) parallel transported vector. For the case of a

¹²Choose inertial coordinates adapted to the the congruence so that $\bar{S}^\mu = (0, \bar{\mathbf{S}})$, $t^\mu = (0, \hat{\mathbf{t}})$ and $n^\mu = (0, \hat{\mathbf{n}})$. Then (16) amounts to $\frac{d\bar{\mathbf{S}}}{d\tau} = \gamma v (\gamma - 1) [\hat{\mathbf{t}}(\bar{\mathbf{S}} \cdot \frac{\hat{\mathbf{n}}}{R}) - \frac{\hat{\mathbf{n}}}{R}(\bar{\mathbf{S}} \cdot \hat{\mathbf{t}})]$. The expression within the brackets is a vector triple product and we may write it as a double cross product. Letting $\mathbf{v} = v\hat{\mathbf{t}}$ we get $\frac{d\bar{\mathbf{S}}}{d\tau} = \gamma(\gamma - 1) \left(\frac{\hat{\mathbf{n}}}{R} \times \mathbf{v} \right) \times \bar{\mathbf{S}}$, which is a simple equation of three-rotation.

¹³The general idea of the transport equation is most readily understood considering a rigid (non-shearing and non-expanding) reference congruence. When the reference frame rotates – so does the parallel transported vector.

congruence with vanishing rotation (as is appropriate for the hypersurface orthogonal congruence we are here considering) one finds [9]

$$\begin{aligned} \frac{D_{\text{ps}}\bar{S}^\mu}{D_{\text{ps}}\tau} = \bar{S}^\alpha \left[\gamma^2 v (t^\mu \wedge a_\alpha) + (2\gamma^2 - 1)(t^\mu \wedge t^\beta \theta_{\alpha\beta}) + \right. \\ \left. + \gamma v (\gamma - 1) \left(t^\mu \wedge \frac{n_\alpha}{R} \right) \right]. \end{aligned} \quad (18)$$

Here the suffix 'ps' is short for 'projected straight'¹⁴. What (18) tells us is how the stopped spin vector precesses relative to a frame that is spatially parallel transported with respect to the reference congruence.

We can of course also consider the analogue of (18) for the rescaled spacetime. There the congruence acceleration vanishes and we are left with

$$\frac{\tilde{D}_{\text{ps}}\tilde{S}^\mu}{\tilde{D}_{\text{ps}}\tilde{\tau}} = \tilde{S}^\alpha \left[(2\gamma^2 - 1)(\tilde{t}^\mu \wedge \tilde{t}^\beta \tilde{\theta}_{\alpha\beta}) + \gamma v (\gamma - 1) \left(\tilde{t}^\mu \wedge \frac{\tilde{n}_\alpha}{\tilde{R}} \right) \right]. \quad (19)$$

In particular, demanding that the gyroscope should remain pointing in the forward direction t^μ as it follows a spatially straight line in the rescaled spacetime, the left hand side of (19) must vanish and we get

$$0 = \tilde{t}^\alpha (\tilde{t}^\mu \wedge \tilde{t}^\beta \tilde{\theta}_{\alpha\beta}). \quad (20)$$

This equation can be simplified to

$$0 = [\tilde{t}^\beta \tilde{\theta}^\mu{}_\beta]_\perp. \quad (21)$$

As mentioned earlier, (21) holds for all directions \tilde{t}^μ if and only if the congruence is shearfree. So a gyroscope (initially directed in the forward direction) will not precess relative to the forward direction, when transported along a spacetime trajectory whose spatial projection has vanishing curvature relative to the rescaled spacetime, if and only if the congruence is shearfree.

When the congruence is shearfree, (19) is simplified to

$$\frac{\tilde{D}_{\text{ps}}\tilde{S}^\mu}{\tilde{D}_{\text{ps}}\tilde{\tau}} = \tilde{S}^\alpha \gamma v (\gamma - 1) \left(\tilde{t}^\mu \wedge \frac{\tilde{n}_\alpha}{\tilde{R}} \right). \quad (22)$$

Comparing with (18) (setting $a^\mu = 0$ and $\theta^{\mu\nu} = 0$ corresponding to an inertial congruence), we see that if we consider the gyroscope precession with respect to the rescaled spacetime, there is only standard Thomas-precession (see e.g. [9, 16]).

Note that while (22) is a four-vector relation, it is effectively a three-dimensional equation since all the terms are orthogonal to η^μ . Letting $(0, \tilde{\mathbf{v}}) = v\tilde{t}^\mu$, $(0, \tilde{\mathbf{n}}) = \tilde{n}^\mu$ and $(0, \tilde{\mathbf{S}}) = \tilde{S}^\mu$ in coordinates locally comoving with the reference congruence we can express (22) in manifest three-form as (see footnote on previous page for details)

$$\frac{D\tilde{\mathbf{S}}}{Dt} = (\gamma - 1) \left(\frac{\tilde{\mathbf{n}}}{\tilde{R}} \times \tilde{\mathbf{v}} \right) \times \tilde{\mathbf{S}} \quad (23)$$

¹⁴In the coming section we will consider a different derivative connected to a different notion of straightness, that will get the suffix 'ns'.

So here we see how the (rescaled and stopped) spin vector precesses relative to a corresponding frame that is parallel transported with respect to the optical geometry. Note the absence of explicit factors e^Φ and how very simple this law of precession is.

Considering gyroscope precession relative to some curved (rescaled) spatial geometry we must take into account that a parallel transported frame will in general be rotated relative to its initial configuration if we transport it along some closed spatial trajectory. For motion in the equatorial plane of some axisymmetric static optical geometry this can easily be dealt with by introducing a reference frame that rotates relative to a local frame spanned by $\tilde{\mathbf{r}}$ and $\tilde{\boldsymbol{\varphi}}$, in the same manner as a parallel transported reference frame does on a plane. Such a reference frame always returns to its initial configuration after a full orbit. In [9] the effective rotation relative to this new frame of reference is derived. If the line element can be written on the form

$$d\tilde{s}^2 = \tilde{g}_{\tilde{r}\tilde{r}}d\tilde{r}^2 + \tilde{r}^2d\varphi^2 \quad (24)$$

the effective rotation vector can be written as

$$\tilde{\boldsymbol{\Omega}}_{\text{effective}} = (\gamma - 1) \left(\frac{\tilde{\mathbf{n}}}{R} \times \tilde{\mathbf{v}} \right) + \frac{1}{\tilde{r}} \left(\frac{\pm 1}{\sqrt{\tilde{g}_{\tilde{r}\tilde{r}}}} - 1 \right) \tilde{\mathbf{v}} \times \tilde{\mathbf{r}}. \quad (25)$$

We have here included a \pm sign. If $\tilde{\mathbf{r}}$, which is assumed to be pointing away from the center of symmetry, points in the direction of increasing \tilde{r} we have the positive sign, otherwise we should use the negative sign¹⁵. Note that (25), for the particular case of motion in an axisymmetric spatial geometry, gives the precession relative to the 'would-be-flat' reference frame in terms of the parameter time t ¹⁶.

3.2. Conclusion as regards the standard projected curvature

We have seen that the optical geometry, as presented here, retains all of the features listed in the introduction given that the *shear-tensor* of the congruence in question vanishes. This corresponds to a larger set of spacetimes than the conformally static spacetimes, see section 8. We have also seen that gyroscopes precess according to a very simple law of rotation with respect to the optical geometry.

4. Generalizing the optical geometry using a different curvature measure

While vanishing projected curvature is likely to be the first notion of spatial straightness that comes to mind, it is perhaps not the most natural for all cases. In [8], a novel definition is proposed via a variational principle. Let ds be the distance traveled for a test particle as seen from the local congruence observers ($ds = \gamma v d\tau$). Parameterizing

¹⁵For the standard optical geometry of a black hole there is a neck (minimum value of \tilde{r}) at the photon radius. For this geometry we should use the positive sign outside of the photon radius and the negative sign inside of this radius. Note that $\frac{1}{\sqrt{\tilde{g}_{\tilde{r}\tilde{r}}}} = 0$ at the photon radius, so there is no discontinuity in $\tilde{\boldsymbol{\Omega}}_{\text{effective}}$.

¹⁶This time equals the local time of the congruence observers in the rescaled spacetime. In the case of standard optical geometry for a Schwarzschild black hole, it is simply the Schwarzschild time.

the trajectory by λ , the integrated distance δs along a trajectory connecting two fixed spacetime points can be written as

$$\delta s = \int ds \quad (26)$$

$$= \int \sqrt{P_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda. \quad (27)$$

One can show [8] that in order for this action to be stationary (minimized) with respect to variations of the trajectory that are perpendicular to η^μ and t^μ , the projected curvature must obey

$$v \frac{n_\mu}{R} = -2[t^\alpha \theta_{\alpha\mu}]_\perp. \quad (28)$$

Trajectories that are obeying this relationship are thus minimizing¹⁷ the spatial distance traveled and are said to be straight in the new sense or simply new-straight (for want of a better name). Thus the two notions of straightness differ (in general) if and only if there is shear. Notice that the curvature relation of (28) is velocity dependent. For more details and some intuition of why the two notions of straightness differ see [8].

One may introduce a new curvature measure from how fast a trajectory deviates from a corresponding (same v and t^μ) new-straight trajectory. Denoting the corresponding curvature direction and curvature radius by \tilde{n}^μ and \tilde{R} respectively, the inertial force formalism [8], in an optically rescaled spacetime (analogous to the outline in the preceding section) takes the form

$$\begin{aligned} \frac{1}{m\gamma^2} e^\Phi \left(\frac{F_\parallel}{\gamma} \tilde{t}^\mu + F_\perp \tilde{m}^\mu \right) &= \frac{1}{\gamma^2} \tilde{P}^{\mu\rho} \tilde{\nabla}_\rho \Phi + \frac{v}{\gamma^2} (\tilde{t}^\alpha \tilde{t}^\beta \tilde{\theta}_{\alpha\beta} + \tilde{\eta}^\rho \tilde{\nabla}_\rho \Phi) \tilde{t}^\mu \\ &+ \frac{dv}{d\tilde{\tau}_0} \tilde{t}^\mu + v^2 \frac{\tilde{n}^\mu}{\tilde{R}}. \end{aligned} \quad (29)$$

It follows immediately that a geodesic photon (set the left hand side to 0 and $v = 1$) has zero curvature ($1/\tilde{R} = 0$) in the new sense relative to the rescaled spacetime. Also, the sideways force on a massive particle following a straight line is independent of the velocity. Notice that this holds independent of whether the congruence is shearing or not. So in fact, with the new sense of curvature, for *any* spacetime and *any* spacelike foliation (and corresponding rescaling), a geodesic photon follows a spatially straight line, and the sideways force on an object following a spatially straight line is independent of the velocity.

4.1. Gyroscope precession

In [9], a spin precession formalism connected to the new-straight curvature measure is presented, analogous to (18) above for the projected curvature measure. Relative to the rescaled spacetime (set $a^\mu = 0$, $\omega^{\mu\nu} = 0$ and put tilde on everything) we have from [9]

$$\frac{\tilde{D}_{\text{ns}} \tilde{S}^\mu}{\tilde{D}_{\text{ns}} \tilde{\tau}} = \tilde{S}^\alpha \left[-(\tilde{t}^\mu \wedge \tilde{t}^\beta \tilde{\theta}_{\alpha\beta}) + \gamma v (\gamma - 1) \left(\tilde{t}^\mu \wedge \frac{\tilde{n}_\alpha}{\tilde{R}} \right) \right]. \quad (30)$$

¹⁷Strictly speaking, the distance traveled with respect to the congruence observers is stationary with respect to variations perpendicular to η^μ and t^μ if the curvature obeys (28).

Here 'ns' stands for new-straight¹⁸. While the term containing the expansion-shear tensor is simpler in this equation, compared to (18), it is not vanishing. Thus, analogous to the discussion in section 3.1, a gyroscope initially directed in the forward direction will remain in the forward direction, as we move along an arbitrary line that is straight in the new sense relative to the rescaled spacetime, if and only if the shear-tensor of the congruence vanishes.

4.2. Fermat's principle and the new-straight curvature

As discussed in [8], the new-straight curvature relative to the rescaled spacetime is closely related to Fermat's principle. Indeed for a photon in the rescaled spacetime, the coordinate time it takes for a photon to go from a certain event along one spatial point (congruence line) to another spatial point (congruence line) is given by

$$\delta t = \int dt = \int d\tilde{s}. \quad (31)$$

Fermat's principle states that a null trajectory is a geodesic if and only if it extremizes δt ¹⁹. Also, by *definition* $\int d\tilde{s}$ is extremized²⁰ if and only if the curvature in the new sense (with respect to the rescaled spacetime) vanishes, which according to (31) means that δt is extremized. It follows that any null geodesic has vanishing spatial curvature in the new sense relative to the rescaled spacetime.

The connection between Fermat's principle and straight lines in the optical geometry was realized, for conformally static spacetimes, a long time ago. With the new definition of curvature the connection holds in any spacetime.

4.3. Conclusion regarding the new sense of curvature in relation to optical geometry

We have seen that with the new sense of curvature, photons move along spatial geodesics for any slicing and corresponding rescaling in any spacetime. Also the sideways force on a particle following a spatially straight line will be independent of the spatial velocity. Gyroscopes following straight lines will however precess (in general) relative to the forward direction.

5. Some comments on rescalings and other transformations

We have in this article used conformal rescalings more or less without motivation. In principle one can imagine other transformations. In particular we can consider a transformation of just the spatial geometry, and do this in such a manner that no

¹⁸Note that the bar in \tilde{n}_α and the bar in \tilde{R} have nothing to do with the bar in \tilde{S}^α .

¹⁹We here naturally refer to the part of the null trajectory that connects the two congruence lines in question. By extremizes we mean that we may have a minimum or a saddlepoint in δt with respect to (null-preserving) variations around the trajectory in question.

²⁰Strictly speaking $\int d\tilde{s}$ is extremized with respect to variations perpendicular to the trajectory. For the case of null-preserving variations, perpendicular variations are however (to the necessary order) the only allowed variation.

matter what shear or rotation the reference congruence has, all the projected geodesic photons get vanishing spatial curvature. But in fact, this is not generally doable. To see this, consider the projection of a left-moving and a right-moving geodesic photon when the congruence rotates (think in 2+1 inertial coordinates). The projected curvature direction (relative to the standard on-slice geometry) will be *opposite* for the two directions as illustrated in figure 1.

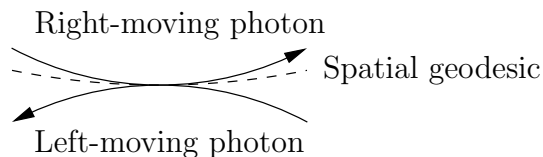


Figure 1. Photons moving in opposite spatial directions will in general have different projected curvatures when the congruence is shearing or rotating. This means that the motion of photons cannot generally correspond to projected geodesics of any three-geometry for a rotating or shearing congruence.

The two projected trajectories are aligned at the origin, but are then separating. A geodesic aligned with the two trajectories can however not split in two. Hence there simply exists *no* three-geometry relative to which the projections of general geodesic photons will correspond to spatial geodesics if the congruence is rotating. The same argument applies when the congruence is shearing. So whatever transformation of the spatial metric we are considering, using the standard projected curvature, the optical congruence would have to be rotationfree and shearfree for geodesic photons to follow spatial geodesics.

But why conformal rescalings? Well, as we argued before, we need a congruence and a corresponding orthogonal slicing to get an isotropic speed of light (with respect to coordinate time), no matter what spatial metric we come up with. Also, for the chosen labeling of the time-slices, if we want the speed of light (with respect to the transformed) spatial metric to be unit everywhere ($\frac{d\tilde{s}}{dt} = 1$), where $d\tilde{s}$ is the transformed spatial distance (no matter what transformation we are considering), then a rescaling of spatial distances by a factor $e^{-\Phi}$ is in fact the only option. That geodesic photons follow projected straight lines relative to the rescaled space, we may see as a pure bonus when there is no shear. Notice however that even if we would relax the unit speed requirement, we would still need a shearfree congruence to get photons to follow straight spatial lines in the projected sense.

In summary, as concerns the unit speed requirement, the rescaling is the only option. Concerning the requirement that the projection of a null geodesic should correspond to a spatially straight line, no other transformation would make a better job²¹. Also, using the new-straight curvature, we get *all* of the features concerning photons that we want using the rescaling scheme.

²¹In other words no other transformation would be applicable to a larger set of spacetimes (and congruences) than those already considered for the generalized (in the projected sense) optical geometry.

Notice that (as we have argued above) the fact that photon geodesics are unaffected by the conformal rescaling, is not of any *direct* importance. Considering the projected curvature, we can in principle imagine some other spacetime transformation that does affect null geodesics, but that gives a vanishing projected curvature (in the transformed spacetime) for a photon that is a geodesic in the original spacetime. However, for conformally static spacetimes, assuming that we rescale away time dilation (lapse), the fact that photon geodesics are unaffected by the rescaling *trivially* means that the projected spatial curvature of a geodesic photon must vanish²².

Notice also that whatever slicing we choose in a general spacetime, we get a corresponding rescaled spacetime with a line element of the form $\text{BlockDiag}[-1, g_{ij}(t, \mathbf{x})]$, where the motion of free photons corresponds to null geodesics. In the rescaled spacetime the only degrees of freedom are those of the spatial metric. So from this point of view, we can always have a space that accounts for the behavior of photons. Indeed the formalism of the new-straight curvature can be seen as a certain way of describing *how* that space dictates the motion of geodesic photons.

6. The generalized optical geometry in coordinates, assuming vanishing shear

In order to get a better feeling for the properties of spacetimes that admits a hypersurface-forming shearfree vector field, it is instructive to carry out an analysis in coordinates adapted to the chosen time slices and the corresponding orthogonal congruence. In such coordinates, the rescaled metric takes the following form

$$\tilde{g}_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & h_{ij} \end{bmatrix}. \quad (32)$$

In a spacetime of the form (32) the only non-zero elements of the affine connection are

$$\tilde{\Gamma}_{ij}^0 = \frac{1}{2} \partial_t h_{ij} \quad (33)$$

$$\tilde{\Gamma}_{i0}^k = \frac{1}{2} h^{kl} \partial_t h_{li} = \tilde{\Gamma}_{0i}^k \quad (34)$$

$$\tilde{\Gamma}_{ij}^k = \frac{1}{2} h^{kl} (\partial_i h_{lj} + \partial_j h_{il} - \partial_l h_{ij}) \equiv \tilde{\gamma}_{ij}^k. \quad (35)$$

Using these relations it is easy to evaluate the shear-expansion tensor in the coordinates in question

$$\begin{aligned} \tilde{\theta}_{ij} &= \frac{1}{2} (\tilde{\nabla}_i \tilde{\eta}_j - \tilde{\nabla}_j \tilde{\eta}_i) = \dots \\ &= \frac{1}{2} \partial_t h_{ij}. \end{aligned} \quad (36)$$

Here Latin indices run from 1 to 3. The other components of $\tilde{\theta}_{\alpha\beta}$ are zero. As mentioned before, the shear tensor vanishes if and only if $[\tilde{\theta}^\mu{}_\alpha \tilde{t}^\alpha]_\perp = 0$ for all \tilde{t}^μ . Hence, using (36)

²²Thinking of geodesics as maximizing the proper time (null geodesics being a limiting case), it is obvious that in the absence of time dilation (lapse), shear and rotation, a geodesic must locally take the shortest spatial path, thus having vanishing projected curvature.

(raising the first free index to get $\tilde{\theta}^i_j = \frac{1}{2}h^{ik}\partial_t h_{kj}$), necessary and sufficient conditions for the congruence shear to vanish (in the coordinates in question) is

$$h^{ik}\partial_t h_{kj} \propto \delta^i_j. \quad (37)$$

Multiplying both sides by h_{li} yields

$$\partial_t h_{lj} \propto h_{lj}. \quad (38)$$

What this means is that when moving in time only, all the components of h_{ij} must increase with the same factor, for every fixed \mathbf{x} . The most general form for h_{ij} is then

$$h_{ij} = e^{2\Omega(t,\mathbf{x})}\bar{h}_{ij}(\mathbf{x}). \quad (39)$$

Here Ω is an arbitrary well behaved function of \mathbf{x} and t , and $\bar{h}_{ij}(\mathbf{x})$ is independent of the time coordinate. Using this in (32) yields

$$\tilde{g}_{\mu\nu} = \begin{bmatrix} -1 & 0 \\ 0 & e^{2\Omega(t,\mathbf{x})}\bar{h}_{ij}(\mathbf{x}) \end{bmatrix}. \quad (40)$$

So, in coordinates adapted to the time slices and corresponding congruence, the line element takes the above form if and only if the congruence is shearfree. The conformally static spacetimes is a subset (set $\Omega = 0$) of the spacetimes described by conformal rescalings of (40).

Incidentally one may use the above coordinate formalism to verify the necessary and sufficient condition of vanishing shear in order for (null) geodesics in the rescaled spacetime to correspond to projected straight lines²³.

6.1. The inertial force formalism in coordinates

Inserting $h_{ij} = e^{2\Omega(t,\mathbf{x})}\bar{h}_{ij}(\mathbf{x})$ in (36) we find

$$\tilde{\theta}_{ij} = (\partial_t \Omega)h_{ij}. \quad (41)$$

Using this relation in (29) we readily find the optical inertial force formalism in coordinates adapted to the congruence, assuming vanishing shear

$$\frac{e^\Phi}{m} \left(\frac{F_{\parallel}}{\gamma} \tilde{t}^k + F_{\perp} \tilde{m}^k \right) = h^{kl} \nabla_l \Phi + v \tilde{t}^k \partial_t (\Phi + \Omega) + \gamma^2 \frac{dv}{dt} \tilde{t}^k + \gamma^2 \frac{v^2}{R} \tilde{n}^k. \quad (42)$$

Setting $\partial_t \Omega = 0$, yields the inertial force equation in the standard optical geometry for conformally static spacetimes. Also setting $\partial_t \Phi = 0$ yields the inertial force equation in standard optical geometry for static spacetimes.

Multiplying (42) by m and shifting the first two terms on the right hand side to the left, we may identify two different types of inertial forces as

$$\begin{aligned} \text{Gravity} & : -mh^{kl}\nabla_l\Phi \\ \text{Expansion} & : -m(\partial_t\Phi + \partial_t\Omega)v^k. \end{aligned}$$

²³The three spatial equations for a geodesic in the rescaled spacetime, in the coordinates in question, takes the form of $\frac{d^2x^k}{d\tilde{\tau}^2} + \tilde{\gamma}_{ij}^k \frac{dx^i}{d\tilde{\tau}} \frac{dx^j}{d\tilde{\tau}} = -2\tilde{\Gamma}_{i0}^k \frac{dt}{d\tilde{\tau}} \frac{dx^i}{d\tilde{\tau}}$. Since $\tilde{\gamma}_{ij}^k$ is the affine connection on the slice, it is obvious that we must have $\tilde{\Gamma}_{i0}^k \propto \delta^k_i$, in order for the projected trajectory to correspond to a spatial geodesic. According to the above discussion, this holds if and only if the congruence shear vanishes.

The term 'Gravity' is introduced by analogy to the Newtonian sense of gravity. From the point of view of general relativity, this term would simply be called acceleration (referring to the congruence acceleration). Strictly speaking, the term 'Expansion' refers to expansion in the non-rescaled spacetime²⁴. For positive $\partial_t(\Phi + \Omega)$ the term has the form of a viscous damping force although for negative $\partial_t(\Phi + \Omega)$ it is rather a velocity proportional driving force. The last two terms of (42) are a representation of the motion (acceleration) relative to the reference congruence, and are not regarded as inertial forces. For further discussion of these types of interpretations, see [8]. Note however that precisely what we call an inertial force is ambiguous up to factors of γ . We could for instance multiply the entire equation, or just the parallel part of the equation, by γ and thus introduce γ -factors in the inertial forces.

7. A note on given and received forces

In the formalism thus far presented, we have expressed forces in terms of what is experienced by an observer comoving with the test particle in question. These forces are in general different from the forces needed to be *given* by the congruence observers, in order to make the test particle move as specified by the curvature, curvature direction and the time derivative of the speed. In [8] the relationship between the given forces $F_{c\perp}$ and $F_{c\parallel}$ (where 'c' stands for congruence) and the received (comoving) forces F_{\perp} and F_{\parallel} is derived

$$F_{c\perp} = \frac{F_{\perp}}{\gamma} \tag{43}$$

$$F_{c\parallel} = F_{\parallel} \tag{44}$$

These relations can be used to get the given forces in any of the different formulations of this article ((11),(29) and (42)). For instance we may consider a railway track in some static geometry. The perpendicular force exerted by the track (as seen from the rest frame of the track) on the train is given by $F_{c\perp} = \frac{F_{\perp}}{\gamma}$. Note that if the track has vanishing optical curvature, this force (unlike the comoving force) has a velocity dependence. Indeed it is easy to find [11] that there is no way to lay out a railway track (no curvature) such that the sideways *given* force is velocity independent.

8. A note on uniqueness

In the scheme of section 4 (employing the novel curvature measure), we can do optical geometry in a finite region around any point in any spacetime. There are always more than one possible such geometry (corresponding to different choices of slices).

The optical geometry as generalized in section 3 (using the projected curvature) is however more restrictive and the standard optical geometry is more restrictive still. In

²⁴It is easy to show, for instance comparing (42) with (11) and using the formulas for how the kinematical invariants are transformed as listed in [8], for the particular case of vanishing shear, that we have $\partial_t(\Phi + \Omega) = e^{\Phi} \frac{1}{3} \theta$.

the coming two subsections we comment on the similarities and differences between the standard and the generalized optical geometry of section 3.

8.1. The standard optical geometry

Standard optical geometry is defined for conformally static spacetimes. These are defined as spacetimes that admits a timelike hypersurface forming conformal Killing field. Mathematically this amounts to that there must exist a scalar field f and a vector field ξ^μ that obeys²⁵

$$(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = f g_{\mu\nu} \quad (45)$$

$$P_\mu{}^\rho P_\nu{}^\sigma (\nabla_\sigma \xi_\rho - \nabla_\rho \xi_\sigma) = 0 \quad (46)$$

Here $P_\mu{}^\rho = \delta_\mu{}^\rho + \frac{1}{-\xi^\sigma \xi_\sigma} \xi_\mu \xi^\rho$. The first equation is the conformal Killing equation. The second equation corresponds to setting $\omega_{\mu\nu}$ as defined in (5), to zero²⁶. For the particular case when $f = 0$ we have a Killing field rather than a conformal Killing field. Given a field ξ^μ that obeys (45) and (46), we define $e^{2\Phi} = -g_{\alpha\beta} \xi^\alpha \xi^\beta$. After a rescaling $\tilde{g}_{\mu\nu} = e^{-2\Phi} g_{\mu\nu}$ we get our optical geometry. Notice however that for any solution (ξ^μ, f) , we can form another solution as $(\alpha \xi^\mu, \alpha f)$ for some constant α . Thus the definition of $e^{2\Phi}$ is not unique. For simple cases, like a Schwarzschild black hole, where we have asymptotic flatness, we can however choose α so that ξ^μ is normalized at infinity.

There can however be a larger freedom still in ξ . For instance in flat spacetime, in inertial coordinates, *any* normalized timelike constant vector field satisfies the requirements (although the optical geometry is flat for all choices). There are however also other, not quite so trivial, timelike hypersurface forming Killing fields for a flat spacetime. In particular there is a Killing field parallel to the four-velocities of the points of a rigidly accelerating system (a so called Rindler system[12]). The associated optical geometry is here curved. Note that this Killing field is not a global field however.

In summary, there may exist no standard optical geometry, and there may exist more than one (non-trivially related) standard optical geometry, depending on the spacetime in question.

8.2. The generalized optical geometry for the projected curvature

In the generalized optical geometry of section 3 (using the projected curvature), the fundamental equation (requirement) is not really an equation for a vector field, but for a congruence of worldlines. There has to exist a non-rotating, non-shearing congruence for this type of generalized optical geometry to work. This is however equivalent to the existence of a rotationfree and shearfree vector field ζ^μ (not necessarily normalized) that obeys

$$P_\mu{}^\rho P_\nu{}^\sigma (\nabla_\rho \zeta_\sigma + \nabla_\sigma \zeta_\rho) = f P_{\mu\nu} \quad (47)$$

$$P_\mu{}^\rho P_\nu{}^\sigma (\nabla_\sigma \zeta_\rho - \nabla_\rho \zeta_\sigma) = 0 \quad (48)$$

²⁵Assuming four dimensions, it follows from (45) that $f = \frac{1}{2} \nabla_\alpha \xi^\alpha$.

²⁶Substitute η^μ by ξ^μ , and modify the form of $P_\mu{}^\rho$ as was just shown.

Here $P_\mu{}^\rho = \delta_\mu{}^\rho + \frac{1}{-\zeta^\sigma \zeta_\sigma} \zeta_\mu \zeta^\rho$ and again f is some scalar function²⁷. The first equation corresponds to setting $\sigma_{\mu\nu}$ as defined in (4) to zero²⁸. The second equation corresponds to setting $\omega_{\mu\nu}$ as defined in (5) to zero²⁹. Expressed in this form we note that the only difference from the standard optical geometry equations (substituting ζ^μ with ξ^μ) lies in the projection operators in (47). Indeed (47) is the projected version of (45). This means that the standard optical geometry equations are more restrictive than the latter two equations. Any field that satisfies (45) and (46), will also satisfy (47) and (48), but the converse is not generally true.

So, given a field ζ^μ that satisfies (47) and (48) the generalized optical geometry exists. We cannot however in general find the rescaling parameter (modulo a single constant) from the norm of ζ^μ ³⁰. To find the rescaling we would instead consider the slices orthogonal to the field and assign a continuous parameter t to these slices. We would then make a rescaling so that $\tilde{g}^{\alpha\beta} \nabla_\alpha t \nabla_\beta t = -1$. We may understand that the freedom in the labeling t of the time slices, gives a freedom of a scale factor (as a function of t) for the whole (spatial) optical geometry. For any standard optical geometry there are thus a multitude of generalized optical geometries, for the same reference congruence. As we will demonstrate in a forthcoming paper [13] one can also have a standard optical geometry connected to a certain reference congruence, and in the same region of spacetime have a generalized optical geometry for a different congruence.

More importantly however, there are cases where there exists no standard optical geometry, but where the generalized optical geometry exists. In the companion paper [13] we illustrate that there exists a generalized optical geometry (within a finite region of any spacetime point) for any spherically symmetric spacetime. Indeed there are infinitely many (in-falling) non-shearing reference congruences for this case. In particular we show that there exists a generalized optical geometry across the horizon of a Schwarzschild black hole (unlike in the standard optical geometry).

The equations (47) and (48) are here mainly included for comparison with the standard optical geometry equations. In the companion paper [13], we in fact find it more convenient to work in coordinates, starting from (40).

9. Summary and conclusion

A generalization of the optical geometry has been proposed prior to this paper [4] using a different philosophy, but see [14] for criticism. The optical geometry as presented in this article can be done at different levels.

²⁷In four dimensions it follows that $f = \frac{2}{3} P^{\alpha\beta} \nabla_\alpha \zeta_\beta$.

²⁸Substitute ζ^μ by ξ^μ , and modify the form of $P_\mu{}^\rho$ as was just shown. Note that when applying (4) to a non-normalized vector field, the appropriate θ (corresponding to $\frac{3}{2}f$) is given by $\theta = P^{\alpha\beta} \nabla_\alpha \zeta_\beta$.

²⁹Substitute ζ^μ by ξ^μ , and modify the form of $P_\mu{}^\rho$ as was just shown.

³⁰In fact (47) and (48) are independent of the norm of ζ^μ in the sense that if ζ^μ solves (47) and (48) so will $h\zeta^\mu$ where h is some arbitrary function. This is contrary to (45) of the preceding section, where given a field ξ^μ that solves (45), a field $h\xi^\mu$ would solve (45) if and only if h was constant.

- In any spacetime we can produce an optical geometry where photons move with unit speed. Using the new sense of curvature [8], a geodesic photon follows a spatially straight line and the sideways force on a massive particle following a straight line is independent of the velocity.
- In any spacetime that admits a hypersurface orthogonal shearfree congruence of worldlines, we can produce an optical geometry where photons move with constant speed. Here the projected curvature and the new-straight curvature coincide and therefore either one can be used to define what is spatially straight. A geodesic photon follows a spatially straight line and the sideways force on a massive particle following a straight line is independent of the velocity. Furthermore, a gyroscope initially pointing in the direction of motion, following a spatially straight line, will not precess (independently of the velocity) relative to the forward direction. If the gyroscope follows a spatially curved line, it obeys a very simple law of three-dimensional precession (23).
- In a conformally static spacetime (choosing the preferred congruence) we have the same features of the optical geometry as outlined in the preceding point. Here the optical geometry is static. Furthermore, as demonstrated in [15], Maxwell's equations take a simple form written in terms of the optical metric.

In any generalization of a theory, there are in general several possibilities. Indeed if we consider the standard optical geometry, in conformally static spacetimes, we may use either the projected or the new-straight curvature (they are identical here). Generalizing to more complicated spacetimes we however get two different theories (at least algebraically) depending on what curvature measure we are adopting. One may certainly consider other ways of defining an optical geometry. For instance, one might consider relaxing the spatial geodesic requirement of photons and replace it with some preferably simple law (indeed that is what we get if we consider the projected curvature when there is shear, see (11)).

Apart from extending the set of spacetimes where one can introduce optical geometry, this article also aims to present a solid inertial force formalism both for the generalized and the standard optical geometry (again see [14] for criticism of previous works).

As regards applications of the generalized optical geometry we refer to a companion paper [13] where a non-static (but shearfree) congruence is employed to do optical geometry across the horizon of a static black hole.

Appendix A. The Fermi-Walker equation for the gyroscope spin vector in the rescaled spacetime

Consider a rescaled spacetime $\tilde{g}_{\mu\nu} = e^{-2\Phi}g_{\mu\nu}$. Let k^μ be a general vector defined along a worldline of four-velocity v^μ . Introducing $\tilde{v}^\mu = e^\Phi v^\mu$ and $\tilde{k}^\mu = e^\Phi k^\mu$ one may show [8] that the relation between the rescaled and the non-rescaled covariant derivative of the

vector k^μ is given by

$$\frac{\tilde{D}\tilde{k}^\mu}{\tilde{D}\tilde{\tau}} = e^{2\Phi} \left(\frac{Dk^\mu}{D\tau} - (v^\mu k^\rho - v^\alpha k_\alpha g^{\mu\rho}) \nabla_\rho \Phi \right). \quad (\text{A.1})$$

In particular, for $k^\mu = v^\mu$, we get the transformation of the four-acceleration

$$\frac{\tilde{D}\tilde{v}^\mu}{\tilde{D}\tilde{\tau}} = e^{2\Phi} \left(\frac{Dv^\mu}{D\tau} - (v^\mu v^\rho + g^{\mu\rho}) \nabla_\rho \Phi \right). \quad (\text{A.2})$$

The Fermi-Walker equation for a gyroscope of spin vector S^μ is given by

$$\frac{DS^\mu}{D\tau} = v^\mu S_\alpha \frac{Dv^\alpha}{D\tau}. \quad (\text{A.3})$$

Using (A.1), substituting $k^\mu \rightarrow S^\mu$, and (A.2) together with the $S_\alpha v^\alpha = 0$ we readily find

$$\frac{\tilde{D}\tilde{S}^\mu}{\tilde{D}\tilde{\tau}} = \tilde{v}^\mu \tilde{S}_\alpha \frac{\tilde{D}\tilde{v}^\alpha}{\tilde{D}\tilde{\tau}}. \quad (\text{A.4})$$

Thus a spin vector that is Fermi-Walker transported relative to the non-rescaled spacetime is Fermi-Walker transported also relative to the rescaled spacetime if we just rescale the spin vector itself. The converse obviously holds also.

Appendix B. Shearfree congruences

Assuming that $[\tilde{\theta}^\mu{}_\beta \tilde{t}^\beta]_\perp = 0$, for all directions \tilde{t}^μ we have

$$\tilde{\theta}^\mu{}_\beta \tilde{t}^\beta \propto \tilde{t}^\mu. \quad (\text{B.1})$$

Knowing that $\tilde{\theta}^\mu{}_\beta = \tilde{\sigma}^\mu{}_\beta + \frac{\tilde{\theta}}{3} \tilde{P}^\mu{}_\beta$ we see that (B.1) is equivalent to $\tilde{\sigma}^\mu{}_\beta \tilde{t}^\beta \propto \tilde{t}^\mu$. We know that $\tilde{\sigma}^\mu{}_\beta \tilde{\eta}^\beta = 0$. Since $\tilde{\sigma}_{\mu\nu}$ is a symmetric tensor it follows that in coordinates adapted to the congruence, only the spatial part of $\tilde{\sigma}_{\mu\nu}$ is nonzero. Also, for (B.1) to hold for arbitrary spatial directions \tilde{t}^i , we must have $\tilde{\sigma}^i{}_j \propto \delta^i{}_j$. Knowing also that the trace $\tilde{\sigma}^\alpha{}_\alpha$ always vanishes, it follows that $\tilde{\sigma}^\mu{}_\nu$ must vanish entirely. This in turn is true if and only if the non-rescaled shear tensor vanishes since $\tilde{\sigma}_{\mu\nu} = e^{-\Phi} \sigma_{\mu\nu}$.

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