Optical geometry across the horizon

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Abstract. In a companion paper (Jonsson and Westman 2006 *Class. Quantum Grav.* **23** 61), a generalization of optical geometry, assuming a non-shearing reference congruence, is discussed. Here we illustrate that this formalism can be applied to (a finite four-volume) of any spherically symmetric spacetime. In particular we apply the formalism, using a non-static reference congruence, to do optical geometry across the horizon of a static black hole. While the resulting geometry in principle is time dependent, we can choose the reference congruence in such a manner that an embedding of the geometry always looks the same. Relative to the embedded geometry the reference points are then moving. We discuss the motion of photons, inertial forces and gyroscope precession in this framework.

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1. Introduction

In [1] it is illustrated how we can generalize the optical geometry (see e.g [2] for an introduction) to a wider class of spacetimes than the conformally static ones. In particular, employing the standard projected curvature (see [1] for alternative curvature measures), the new class of spacetimes consists of those spacetimes that admit a hypersurface forming shearfree congruence of timelike worldlines. We are now curious as to whether any of the standard solutions to Einstein's equations, that are *not* conformally static, falls into the new category. The task is then to look for a congruence such that, in the corresponding coordinates, the metric after rescaling takes the form [1]

$$\tilde{g}_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ & \\ 0 & -e^{2\Omega(t,\mathbf{x})}\bar{h}_{ij}(\mathbf{x}) \end{bmatrix}. \tag{1}$$

Indeed, as will be shown in the coming section, such a congruence can be found (in a finite four-volume of the spacetime) whenever we have spherical symmetry in the original metric. This includes the inside of a Schwarzschild black hole and the horizon as well.

Throughout the article, we will use the timelike $(+,-,-,-)$ convention for the sign of the metric. Also the optical line element will be denoted by $d\tilde{s}$.

2. A spherical line element

A general, time dependent, spherically symmetric line element can be written on the form

$$d\tau^2 = a(r,t)dt^2 - 2b(r,t)dr dt - c(r,t)dr^2 - r^2 d\Omega^2. \tag{2}$$

This we may rewrite as

$$d\tau^2 = r^2 \left(d\bar{\tau}^2 - d\Omega^2\right). \tag{3}$$

Here we have introduced a two-dimensional line element

$$d\bar{\tau}^2 = \frac{a(r,t)}{r^2}dt^2 - \frac{2b(r,t)}{r^2}dr dt - \frac{c(r,t)}{r^2}dr^2. \tag{4}$$

In this reduced spacetime we may introduce an arbitrary timelike initial congruence line. From this line we go a proper orthogonal distance ds to create a new line. From the new line we create yet another line in the same manner. Next we introduce a new spatial coordinate x' that is constant for every congruence line, and where $dx' = ds$. Also introduce time slices of constant t' orthogonal to the congruence lines. The reduced line element then takes the form

$$d\bar{\tau}^2 = f(t',x')dt'^2 - dx'^2. \tag{5}$$

The full line element with respect to these coordinates is then on the form

$$d\tau^2 = r^2 f(t',x') \left(dt'^2 - \frac{1}{f(x',t')}\left(dx'^2 + d\Omega^2\right)\right). \tag{6}$$

Here r is in principle known in terms of x' and t'. After rescaling away the factor $r^2 f(t',x')$, this line element clearly has the form of (1) needed for the generalized optical geometry. The optical geometry is then

$$d\tilde{s}^2 = \frac{1}{f(x',t')}\left(dx'^2 + d\Omega^2\right). \tag{7}$$

So, in whatever spherically symmetric spacetime we consider we can thus do the generalized optical geometry. This includes collapsing stars, the spacetime around the horizon of a Schwarzschild black hole and so forth. Notice however that there is no guarantee for the generalized geometry to work *globally* in these spacetimes. The way we are constructing our congruence, it may for instance go null before we have come very far from our original congruence line. Also, the geometry which will be determined by how we choose our initial congruence line, may be more or less complicated, time dependent and so forth.

2.1. A small note on intuition

We are here considering a congruence that is fixed in the spherical angles. From a dynamical point of view, we have found a radial velocity of infinitesimally separated congruence points such that the proper *shape* that is spanned by the points is preserved, see figure 1.

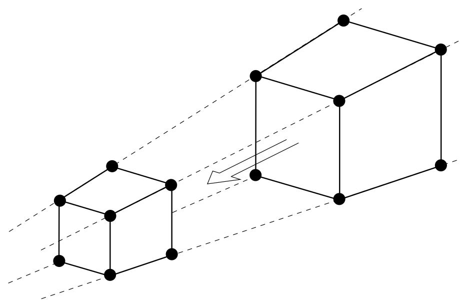

Figure 1. To have vanishing shear (which is necessary for a congruence generating the optical geometry), the congruence points (the black dots), must be shifted in such a way that the shape of a little box of congruence points (as seen when comoving with the box) is the same at all times.

If we start with a cube it must remain a cube, but not necessarily of the same size, in a system comoving with the points. For instance considering a flat space and low velocities inwards (towards the origin of the spherical coordinates) we may understand that the velocity of the inner part of the cube must be smaller than that of the outer part to insure that the cube is not elongated in the radial direction.

3. Optical geometry for a black hole including the horizon

Let us now study a black hole explicitly, with focus on the horizon. The ordinary Schwarzschild coordinates are ill suited for congruences passing through the horizon. There is however another coordinate system (called Painlevé coordinates) in which the Schwarzschild line element is given by

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dT_{\mathrm{P}}^2 - 2\sqrt{\frac{2M}{r}} dT_{\mathrm{P}} dr - dr^2 - r^2 d\Omega^2. \tag{8}$$

This line element is connected to the standard line element through a resetting of the ordinary Schwarzschild clocks[1]. In these coordinates there are no problems in passing through the horizon. We also find it practical to introduce dimensionless coordinates $r/2M \rightarrow x$, $T_{\mathrm{P}}/2M \rightarrow T$. Then the line element takes the form

$$\frac{d\tau^2}{(2M)^2} = \left(1 - \frac{1}{x}\right) dT^2 - 2\frac{1}{\sqrt{x}} dT dx - dx^2 - x^2 d\Omega^2. \tag{9}$$

[1]The clocks are reset in such a way that the coordinate time passed for a freely falling observer initially at rest at infinity corresponds to the proper time experienced by this observer. Inside the horizon one cannot have any material clocks at a fixed r but that doesn't matter.

The reduced line element (compare with (4)) is then given by

$$d\bar{\tau}^2 = AdT^2 - 2BdTdx - Cdx^2. \tag{10}$$

Here the reduced metrical components are given by[2]

$$A = \frac{1}{x^2}\left(1 - \frac{1}{x}\right) \quad B = \frac{1}{x^2}\frac{1}{\sqrt{x}} \quad C = \frac{1}{x^2}. \tag{11}$$

Now we may introduce an arbitrary timelike trajectory that passes the horizon. From this we go a proper distance ds orthogonal to the trajectory, to create a new congruence line and so forth. In general this scheme would completely hide the manifest time symmetry of the black hole. There is however a way to circumvent this as will be shown in the following sections.

4. Keeping the time symmetry, covariant approach

Suppose that we can find an initial trajectory such that the second trajectory (go ds orthogonal to the initial trajectory) is related to the first by a simple translation straight in the T-direction (along the Killing field connected to T). This way we would maintain a certain time symmetry in the optical metric. In figure 2 we see schematically how this would work.

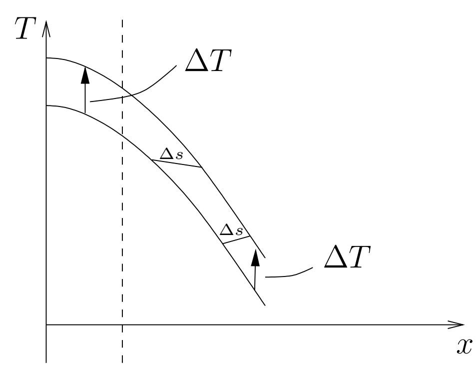

Figure 2. Two congruence lines separated along the Killing field with constant proper distance between them.

Zooming in on the two lines around some specific point, they will to first order be two parallel straight lines. Given the tilt of the lines (i.e. the four-velocity) we can find a relation between the displacement along the Killing field and the orthogonal distance between the lines. How these are related is sketched in Fig 3. Here u^μ is the four-velocity of the trajectories and v^μ is a spacelike vector normed to -1 and orthogonal to u^μ. Just adding vectors we find

$$Kds\xi^\mu = dsv^\mu + \sigma dsu^\mu. \tag{12}$$

[2]One may alternatively use the Eddington Finkelstein coordinates where after corresponding rescalings $A = (1 - 1/x)/x^2$, $B = 1/x^2$, $C = 0$.

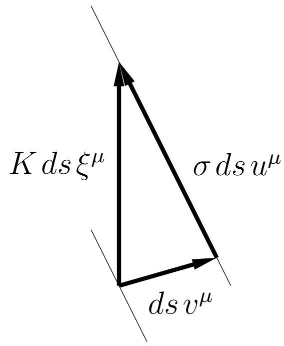

Figure 3. The relation between Killing field ξ^μ, four-velocity u^μ and the orthogonal vector v^μ, assuming an in-going congruence.

Notice that K is a assumed to be a constant unlike σ. Multiplying both sides by u_μ we get

$$\sigma = K\xi^\mu u_\mu. \tag{13}$$

Inserting this back into (12) we get

$$K\xi^\mu = v^\mu + K\xi^\alpha u_\alpha u^\mu. \tag{14}$$

Taking the absolute value of both sides yields shortly

$$K^2 = \frac{1}{(\xi^\alpha u_\alpha)^2 - \xi^\alpha \xi_\alpha}. \tag{15}$$

So K is known given u^μ. Solving for $\xi^\alpha u_\alpha$ yields

$$\xi^\alpha u_\alpha = \pm\sqrt{\xi^\alpha \xi_\alpha + \frac{1}{K^2}}. \tag{16}$$

Outside of the horizon ξ^μ is always timelike and thus the sign in front of the root will be positive[3] . Inside of the horizon, where ξ^μ is spacelike, we can have both signs depending on u^μ. On the horizon we however have $\xi^\mu \xi_\mu = 0$. Thus for finite K we realize that we must have the positive sign on the inside as well to get a continuous four-velocity across the horizon. Using (16) together with $u^\alpha u_\alpha = 1$, we can in principle solve for u^μ given K. The equations are however second order, and there will be four different solutions at every point (see section 5.1 for intuition). There is however a simple way to get first order equations.

Defining v^μ to be the orthonormal vector to u^μ that lies less than 180° clockwise[4] of u^μ, we may write

$$v^\mu = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu} g_{\nu\rho} u^\rho \qquad \text{where} \qquad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

Here $g = -\mathrm{Det}(g_{\mu\nu})$. Then we may rewrite (14) into

$$K\xi^\mu = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu} g_{\nu\rho} u^\rho \pm K\sqrt{\xi^\alpha \xi_\alpha + \frac{1}{K^2}}\, u^\mu. \tag{18}$$

[3]Assuming u^μ and ξ^μ to both be future directed.
[4]In accordance to figure 3, assuming positive values of K, σ and ds.

As noted before, considering an in-falling congruence at the horizon we should choose the positive sign. We see that (18) is a linear equation system which we, given K, should be able to solve to find u^μ. The scheme thus appears successful and there exists an optical (shearfree) congruence that will preserve manifest time symmetry.

5. The congruence for a Schwarzschild black hole

Choosing the positive sign of (18) and assuming a positive K in accordance with the discussion above, (18) takes the form

$$K\xi^\mu = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu}g_{\nu\rho}u^\rho + K\sqrt{\xi^\alpha\xi_\alpha + \frac{1}{K^2}}u^\mu. \tag{19}$$

Assuming the reduced line element to be of the form of (10), and $\xi^\mu = (1,0)$ this can be written

$$K = \frac{B}{\sqrt{g}}u^0 + \frac{C}{\sqrt{g}}u^1 + \sqrt{K^2A+1}u^0 \tag{20}$$

$$0 = \frac{1}{\sqrt{g}}(Au^0 - Bu^1) + \sqrt{K^2A+1}u^1. \tag{21}$$

Recognizing that $dx/dT = u^1/u^0$ we find from the second equation alone that

$$\frac{dx}{dT} = \frac{A}{B - \sqrt{g}\sqrt{K^2A+1}}. \tag{22}$$

Inserting the metrical components of (11) into (22) we find

$$\frac{dT}{dx} = \frac{\frac{1}{\sqrt{x}} - \sqrt{\frac{K^2}{x^2}(1-\frac{1}{x})+1}}{1-\frac{1}{x}}. \tag{23}$$

So here we have the tilt of the reference congruence lines in the Painlevé coordinates. For an infinite value of the free parameter K, outside the horizon, this corresponds to a congruence at rest (i.e. the classical optical congruence). Inside the horizon the root takes a negative value for infinite K and we have no solution.

For any finite values of K we notice that at infinity (23) will correspond to an in-going photon ($\frac{dT}{dx} = -1$). In the particular case of $K = 0$ the congruence will correspond to an in-going photon all the way through the horizon and into the singularity. Photons are however at first sight not particularly well suited for a congruence. For finite K and $x < 1$ we see that the root goes imaginary unless

$$K^2 < \frac{x^3}{1-x}. \tag{24}$$

So the conclusion is that we can do optical geometry, while keeping manifest time symmetry, from a point arbitrarily close to the singularity, across the horizon and all the way towards infinity. Notice in particular that with this scheme we not only get the time symmetry, we insure that we can span the full spacetime all the way towards infinite Schwarzschild times.

5.1. Comments

A short comment may be in order regarding the congruence as we approach infinity, where the spacetime approaches Minkowski. Here one would expect that any congruence with fixed coordinate velocity dx/dt would work as an optical congruence, not just left-moving photons. Indeed from (23) we see that for any large, but finite, x there exists a K such that we can have any in-going coordinate velocity of the congruence. As we go outwards towards infinity from this point the congruence will however start approaching a left moving photon.

Another comment may be in order. The existence of an optical congruence, keeping manifest time symmetry, is independent of what coordinates we are using. In the standard Schwarzschild coordinates it is easy to realize that, both on the inside and the outside, the existence of *one* congruence immediately implies the existence of *another*[5], as depicted in figure 4.

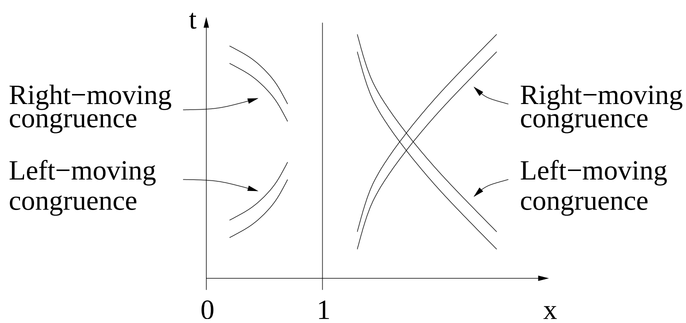

Figure 4. Illustrating in standard (dimensionless) Schwarzschild coordinates that for any left-moving congruence that fulfills the requirements there is also a right-moving congruence that fulfills the requirements.

Inside the horizon this is manifesting itself in the $\pm$ sign of (18). On the outside, where we must have a plus in the $\pm$ sign, it manifests itself in the possibility to have negative K. In the latter case we need to consider a slightly different image than that of figure 3, but the mathematics will be identical if we let K assume negative values. In general we may show that

$$\frac{dx}{dT} = \frac{A}{B \mp \sqrt{g}\sqrt{K^2 A + 1}}. \tag{25}$$

The minus in this case corresponds to a left-moving congruence, and the plus a right-moving.

[5]Except if the congruence on the outside would be parallel to the Killing field, or equivalently if the congruence on the inside would be perpendicular to the Killing field.

6. The optical metric

Let the spatial coordinate difference dx', separating two nearby congruence lines, equal the proper orthogonal distance ds between the lines[6]. Let the new coordinate time difference dt', separating two time slices, equal the original coordinate difference dT, as measured along the Killing field. The reduced metric takes a new form according to

$$d\bar{\tau}^2 = AdT^2 - 2BdTdx - Cdx^2 \qquad \rightarrow \qquad d\bar{\tau}^2 = f(x',t')dt'^2 - dx'^2. \tag{26}$$

Here f is a function yet to be determined. Recall the relation between the various vectors, as depicted in figure 5.

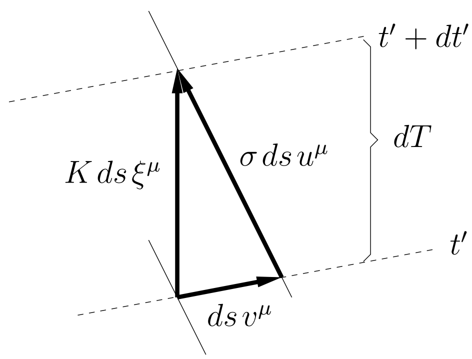

Figure 5. The relation between Killing field, four-velocity and the orthogonal vector. The dotted lines are the new local time slices

Like before we assume the Killing field to be $(1, 0)$ so that $dT = Kds = dt'$. The proper distance squared, as measured along a congruence line, separating two time slices can be expressed as

$$d\bar{\tau}^2 = (\sigma ds)^2 u^\mu u_\mu \qquad d\bar{\tau}^2 = f dt'^2. \tag{27}$$

From (13) and (16) respectively we have

$$\sigma = K\xi^\alpha u_\alpha \qquad \xi^\alpha u_\alpha = \sqrt{A + \frac{1}{K^2}}. \tag{28}$$

Like before we have chosen the positive sign of the root. Putting the pieces together we find

$$f = A + \frac{1}{K^2}. \tag{29}$$

The total, original, line element in the new coordinates is now given by

$$d\tau^2 = (2M)^2 x^2 \left(f(x)dt'^2 - dx'^2 - d\Omega^2 \right). \tag{30}$$

Using (11) and (29), the optical metric is thus given by:

$$d\tilde{s}^2 = \frac{1}{\frac{1}{x^2}\left(1 - \frac{1}{x}\right) + \frac{1}{K^2}} \left(dx'^2 + d\Omega^2 \right). \tag{31}$$

[6]Recall that the distance between the lines is by definition constant along the lines.

Notice however that it is not in explicit form since we do not have x in terms of x' and t'. We may however recall figure 5 where we for constant x have $dt' = Kds = Kdx'$ (since dx' per definition equals ds). Thus we know that constant x means $dt'/dx' = K$. In the new coordinates we have therefore a schematic picture as depicted in figure 6.

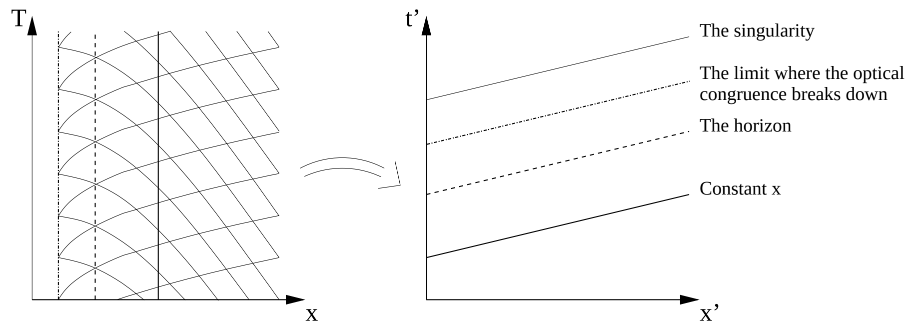

Figure 6. To the left the congruence in the Painlevé coordinates. To the right the horizon etc relative to coordinates adapted to the congruence.

Notice that the Killing field is still a constant vector (tilted up and to the right) in the new coordinates. While we still do not have x analytically in terms of x' and t', we know the qualitative relation well enough to understand some basic features.

6.1. The rubber sheet model

We see from (31) that at spatial infinity the geometry becomes that of a three-cylinder, except if K is infinite. Also we see that on the inner boundary, where the optical congruence breaks down, the stretching (of $d\tilde{s}$ relative to ds) is infinite.

It appears very difficult to do any calculations in our new coordinates, considering that we don't have any explicit relation for x in terms of x' and t'. At every fixed t' we may however express the optical geometry as a function of x, given that we have a relation between dx and dx' (as will be derived in section 7). This background geometry is time independent. The scenario (in 2D) can then be exactly described by a rubber sheet sliding snuggly over the fixed background geometry. Photons move on geodesics with unit velocity at every point if we *comove* with the rubber sheet. The velocity of the rubber sheet will correspond to the velocity of a constant x line relative to the reference congruence. Then we can use our knowledge of geodesics on rotational surfaces, and relative velocities, to find the paths of photons relative this pseudo-optical background geometry.

7. On the relation between x and x'

Given a displacement dx' along the x'-axis we want to find dx. From figure 3 we see that $dx = v^x ds$ or equivalently

$$dx = v^x dx'. \tag{32}$$

From (17) and (20) we readily find

$$v^x = -\alpha u^x \quad \text{where} \quad \alpha = \sqrt{K^2 A + 1}. \tag{33}$$

To find u^x, we solve the linear equation system of (20) and (21) letting $u^0 \to u^T$ and $u^1 \to u^x$. The result is

$$u^x = \frac{K}{\left(\frac{B^2}{\sqrt{g}^2} - (K^2 A + 1)\right)\frac{\sqrt{g}}{A} + \frac{C}{\sqrt{g}}}. \tag{34}$$

Inserting the explicit metrical functions, this miraculously is reduced to

$$u^x = -\frac{x^2}{K}. \tag{35}$$

Using (32) and (33), the general relation between dx and dx' is given by

$$dx' = -\frac{1}{\alpha u^x} dx. \tag{36}$$

In explicit form this is then reduced to

$$dx' = \frac{K}{x^2 \sqrt{\frac{K^2}{x^2}\left(1 - \frac{1}{x}\right) + 1}} dx. \tag{37}$$

Incidentally, using the Eddington-Finkelstein original coordinates yields the same expression, as it must. The expression however turns out not to be particularly easy to integrate analytically except in the limits where K is either infinite or zero. In the limit where K is infinite it however cannot be inverted to find x in terms of x'. In any case (37) is sufficient to express the background geometry explicitly.

8. The background optical geometry

Inserting (37) into (31) we may at a *fix* time t' write the optical line element as

$$d\tilde{s}^2 = \frac{1}{x^4\left(\frac{1}{x^2}\left(1 - \frac{1}{x}\right) + \frac{1}{K^2}\right)^2} dx^2 + \frac{1}{\frac{1}{x^2}\left(1 - \frac{1}{x}\right) + \frac{1}{K^2}} d\Omega^2. \tag{38}$$

In the limit of $K \to \infty$ this takes the familiar form of the standard optical geometry

$$d\tilde{s}^2 = \frac{1}{\left(1 - \frac{1}{x}\right)^2} dx^2 + \frac{x^2}{1 - \frac{1}{x}} d\Omega^2. \tag{39}$$

At the other end, where K goes to zero, it to lowest non-zero order approaches

$$d\tilde{s}^2 = \frac{K^4}{x^4} dx^2 + K^2 d\Omega^2. \tag{40}$$

Here the x-dependence can be taken away by another coordinate transformation. It is then obvious that in this limit we have a flat space. In a symmetry plane this would correspond to a cylinder, infinitely extended in the direction of the singularity but with finite distance from horizon to infinity. Unfortunately in the same limit the optical velocity of constant x position goes to the velocity of light, to lowest order. This means that, if we just concern ourselves with the lowest order influence of K on metrical components and velocities, we will not get any usable dynamics[7]. We can for instance not find the photon radius. If we still would like to use the $K = 0$ limit, we must take higher order terms into account. Perhaps expressions when expanded to the second non-vanishing order in K will be easier to deal with than in the general case. If this would work out it would be no approximation but give the exactly correct dynamics. The point of considering this limit is of course that in this limit the *full* spacetime is spanned by the optical geometry. We will however not pursue this point further here. In any case we may, for arbitrary K, Taylor expand $d\tilde{s}/dx$ in the limit where we approach the innermost point of the optical geometry. Doing this we readily find that regardless of K the momentaneous distance to the innermost point is infinite.

Remember however that the line element of (38) is not strictly the optical geometry. It is not with respect to this element (except in the $K \to \infty$ limit) that photons move on geodesics, as discussed earlier.

The speed $d\tilde{s}/dt'$ of the constant x lines relative to the optical space is easy to derive since we have $K = dt'/dx'$ and from (31) we see that $d\tilde{s} = K dx'/\alpha^2$. Then we find

$$\frac{d\tilde{s}}{dt'} = \frac{1}{\sqrt{\frac{K^2}{x^2}\left(1 - \frac{1}{x}\right) + 1}}. \tag{41}$$

We see that in the limit where $K \to \infty$ this goes to zero as it should. At $K = 0$ it goes to the velocity of light. Incidentally the velocity is at a minimum at $x = 3/2$, the photon radius.

So now we have everything that we need to make explicit calculations in the generalized optical geometry, using the rubber sheet analogy. In fact we may also embed the background optical geometry and visualize the photon radius.

9. Embedding the background geometry

In figure 7 we see a schematic picture of how an embedding of the background geometry would look. Using this qualitative image we will see that the photon radius lies exactly at the neck of the background geometry, just like in standard optical geometry. We will also understand that the way gyroscopes in circular motion precesses, is different inside and outside of the neck.

9.1. Photon geodesics

Study now a photon moving on the surface. At any given radius we can find an angle of the velocity vector of the photon, relative to the rubber sheet, such that instantaneously the photon has no radial velocity relative to the background geometry[8].

[7]Think of the rubber sheet model discussed earlier.

[8]If we direct the photon directly outwards it will move slowly out towards spatial infinity. On the other hand if we direct it purely azimuthally relative to the rubber sheet it will be dragged inwards with the rubber sheet. Somewhere in between there is obviously an angle such that it has purely azimuthal

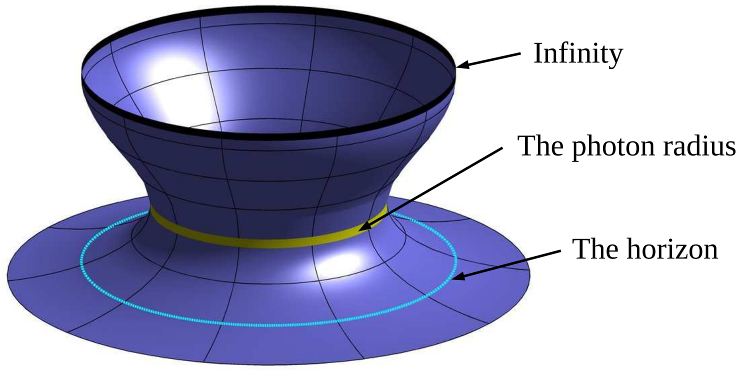

Figure 7. The background optical geometry. The author took the liberty of enhancing the radial variations to get more shape without affecting the qualitative behavior.

A free photon, with an initial position and velocity such that it has no radial velocity, will follow a local geodesic on the surface. However, its position relative to the background will be shifted continously according to the sheet velocity. To time evolve the the position and velocity of the photons, we may however consider the following two-step process. First we move a distance corresponding to the time step dt, along a geodesic on the surface. Then we take the resulting forward direction and parallel transport it downwards corresponding to the shift of the congruence points. In the second step, the angle of the forward direction with a purely radial line will be maintained. This follows from that the congruence is non-shearing (see figure 9 for intuition). We may iterate the two-step process to time evolve the position to arbitrary times.

For a geodesic on a rotational surface it is easy to show that the angle the geodesic makes with a local line of fixed azimuthal angle (a radial line in this case) is *decreasing* with increasing radius, and vice versa. In figure 8 we illustrate the effects of this shift of angles for photon geodesics with no momentary radial velocity.

Looking at figure 8 we may understand that if we are on the outside of the neck, the photon velocity vector will be directed more and more outwards as time goes. Thus it will leave the radius it started at and move to infinity. We also realize that for the corresponding initial velocity vector inside the neck, the velocity vector will be rotated to be directed less and less outwards and thus will start to move inwards. If we start exactly at the neck however, where the embedding radius doesn't change to first order, the photon will remain on the same radius. So, just like in standard optical geometry (see e.g [2]), a geodesic photon in circular motion will stay at the neck of the embedding.

To further clarify the two-step scenario we may plot the evolution as seen from the normal of the surface. This is depicted in figure 9.

Notice that unlike the standard optical geometry (which is a subset of this discussion) we need the velocity of the rubber sheet apart from the background geometry to determine the paths of free photons. We know however that for a given background

velocity relative to the *background*.

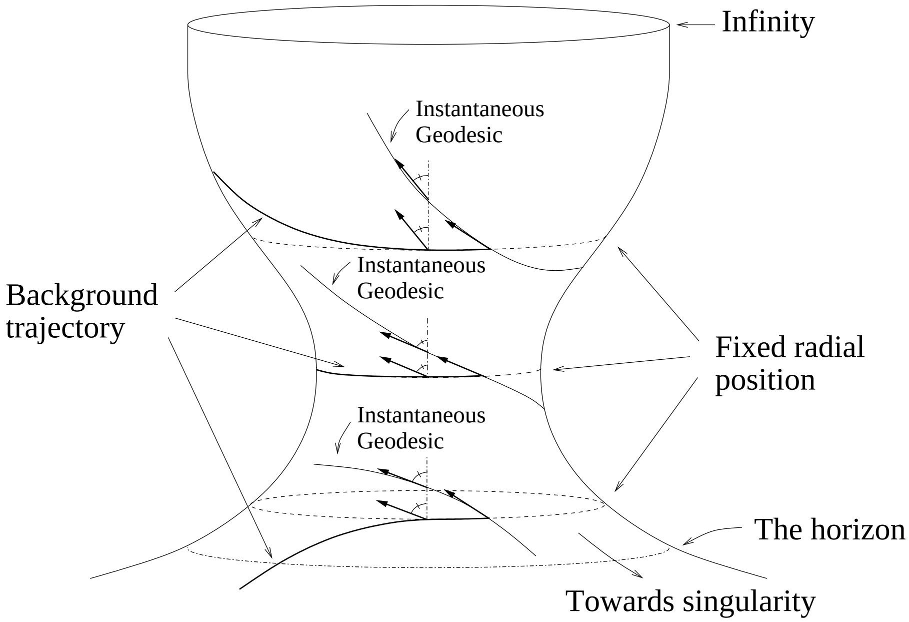

Figure 8. Understanding the photon radius in the generalized optical geometry. We can time evolve the position and velocity of the photon by a two-step process. First we transport the forward direction along a geodesic on the surface and then we parallel transport it downwards according to the shift of the congruence during the time step.

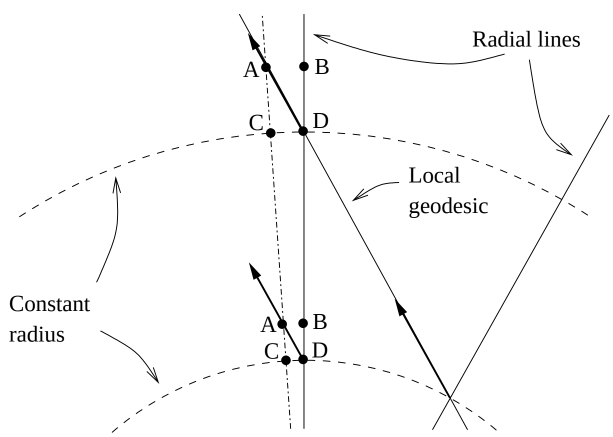

Figure 9. The shift of congruence points A,B,C and D as seen from a local background geodesic coordinate system. Since the congruence is non-shearing, the angle the forward direction makes with a pure radial line is unaltered by the shifting of the congruence points. Notice that the net effect is that the forward direction is parallel transported relative to the background geometry.

geometry the sheet velocity depends only on the embedding radius. The bigger the radius the bigger the velocity. Also, the velocity of the sheet at infinity and at the horizon is that of light. This is sufficient to understand the qualitative behavior of geodesic photons.

9.2. Gyroscope precession for circular motion

Let us now consider circular motion (fixed x) with constant velocity. Consider first motion, outside the photon radius, directed to the left as seen from outside the embedding as illustrated in figure 10.

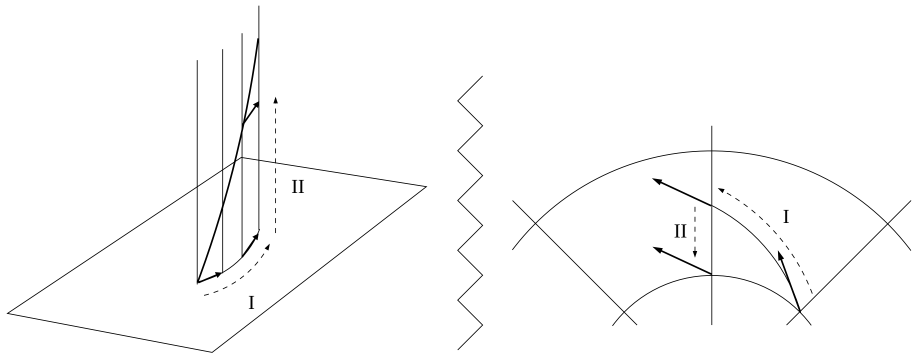

Figure 10. The two step process of moving the forward direction. To the left in 2+1 dimensions, to the right in 2 dimensions.

Moving along a circle of fixed radius with fixed speed, we *know* that after the two-step process of transporting the forward direction, we must get a forward direction that has the same angle relative to the radial line as we had before the two-step process. As is illustrated in figure 10 this means that the optical curvature has to be directed to the left (looking at the surface from the outside).

We know from [1] that a gyroscope undergoes pure Thomas precession relative to the optical geometry. For the case at hand where the trajectory turns left this means that a gyroscope will precess clockwise relative to a corresponding parallel transported vector. Since the forward direction is precessing counterclockwise relative to a parallel transported vector, the gyroscope will precess clockwise relative to the forward direction. It follows that it will precess clockwise relative to the local radial line also. Looking at the embedding from the inside (and from the top), we may say that clockwise circular motion results in counterclockwise precession relative to the forward direction. This is in fact what one expects of gyroscope precession in Newtonian mechanics.

Completely analogously, we may understand that inside the photon radius, looking at the embedding from the top, clockwise circular motion results in *clockwise* precession relative to the forward direction (as seen from the inside looking at the surface). This is not analogous to the Newtonian precession.

Indeed gyroscope precession is easier to deal with in the standard optical geometry (where the congruence is static), but as we have seen it *can* be done also considering an infalling congruence, that allows us to include the horizon.

9.3. Inertial forces considering circular motion

Orbiting a black hole at a fixed radius outside the photon radius requires a smaller outward comoving force the faster one orbits the black hole (like in Newtonian gravity). Inside the photon radius however the required outward force increases the faster one orbits the black hole. This can be readily understood in the standard optical geometry (see e.g [3]) corresponding to infinite K, but when we have a congruence moving relative to the background it is much more complicated to see.

The point is that as we increase the orbital speed, we change the direction of motion relative to the in-falling congruence (tilt the velocity arrow down). This brings about all sorts of changes, for instance the optical curvature (as opposed to the background curvature) changes. This particular feature of black holes apparently cannot be so easily explained, using simple qualitative arguments, when the reference congruence is in-falling.

10. Conclusion

We conclude that the generalized optical geometry (assuming shearfree congruences) can be applied to (a finite sized region of) any spherically symmetric spacetime. In particular we can define an optical geometry from spatial infinity across the horizon and arbitrarily close to the singularity of a static black hole. In 2 spatial dimensions we can display the optical geometry as a curved surface, relative to which the reference congruence points are moving. This motion of the reference points certainly makes any argumentation more complicated than in standard optical geometry. We can however do essentially the same type of qualitative arguments concerning photons, and gyroscopes as in the standard optical geometry, and include the horizon.

References

[1] Jonsson R and Westman H 2006 Generalizing optical geometry *Class. Quantum Grav.* **23** 61-76

[2] Abramowicz M A and Lasota J P 1997 *Class. Quantum Grav.* **14** A23-A30

[3] Abramowicz M A (march) 1993 *Sci. Am.* **266** (no 3) pp 26-31

[4] Jonsson R 2006 Inertial forces and the foundations of optical geometry *Class. Quantum Grav.* **23** 1-36

[5] Jonsson R 2006 A covariant formalism of spin precession with respect to a reference congruence Class. Quantum Grav. **23** 37-59

[6] Abramowicz M A, Nurowski P and Wex N 1995 *Class. Quantum Grav.* **12** 1467-1472

[7] Abramowicz M A 1992 *Mon. Not. R. astr. Soc.* **256** 710-718

[8] Abramowicz MA, Nurowski P and Wex N 1995 *Class. Quantum Grav.* **10** L183-L186

[9] Bini D, Carini P and Jantzen R T 1997 *Int. Journ. Mod. Phys. D* **6** 14