THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Gravity Illustrated

Spacetime Edition

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Cover: An illustration of the curved spacetime for a line through Earth. The small toy car with the pen is used to draw straight lines on the spacetime – corresponding to the motion of three apples thrown out from the center of the earth. The yellow line corresponds to the motion of the yellow apple visible outside Earth. The figure was created using Matlab and Povray.

Back cover: An illustration of the spacetime for a line outside Earth, and the spacetime trajectory of an apple that has been thrown upwards. The figure was created using Matlab.

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Abstract

This thesis deals with essentially four different topics within general relativity: pedagogical techniques for illustrating curved spacetime, inertial forces, gyroscope precession and optical geometry. Concerning the pedagogical techniques, I investigate two distinctly different methods, the *dual* and the *absolute* method.

In the *dual* scheme, I start from the geodesic equation in a 1+1 static, diagonal, Lorentzian spacetime, such as the Schwarzschild radial line element. I then find another metric, with *Euclidean* signature, which produces the same geodesics $x(t)$. This geodesically equivalent *dual* metric can be embedded in ordinary Euclidean space. Freely falling particles correspond to straight lines on the embedded surface.

In the *absolute* scheme, I start from an arbitrary Lorentzian spacetime with a given field of timelike four-velocities u^μ. I then perform a coordinate transformation to the local Minkowski system comoving with the given four-velocity at every point. In the local system the sign of the spatial part of the metric is flipped to create a new metric of Euclidean signature. For the particular case of two dimensions we may embed the absolute geometry as a curved surface. The method is well suited for visualizing gravitational time dilation, cosmological expansion and black holes.

Concerning inertial forces, gyroscope precession and optical geometry, the general framework is based on the introduction of a congruence of reference worldlines in an arbitrary spacetime. This allows us to describe the local motion and acceleration of particles in terms of the speed relative to the congruence, the time derivative of the speed and the spatial curvature (project down along the reference congruence) of the corresponding worldline.

I present two papers concerning inertial forces in this framework, one formal and one intuitive. I also present two papers concerning gyroscope precession, again one formal and one intuitive. In particular I illustrate how one can explain gyroscope precession in an arbitrary stationary spacetime as a double Thomas precession effect.

Introducing a novel type of spatial curvature measure for the worldline of a test particle, we present a natural way of generalizing the theory of optical geometry to include arbitrary spacetimes. The generalized optical geometry allows us to do optical geometry across the horizon of a black hole.

Keywords: curved spacetime, embeddings, pedagogical techniques, inertial forces, gyroscope precession, optical geometry

APPENDED PAPERS

Paper I
Embedding spacetime via a geodesically equivalent metric of Euclidean signature
R. Jonsson – Gen. Rel. Grav. (2001) **33** no 7, pp. 1207-1235.

Paper II
Visualizing curved spacetime
R. Jonsson – Accepted for publication in Am.J.Phys.

Paper III
Inertial forces and the foundations of optical geometry
R. Jonsson – To be submitted to Class. Quant. Grav.

Paper IV
An intuitive approach to inertial forces and the centrifugal force paradox
R. Jonsson – To be submitted to Am.J.Phys.

Paper V
A covariant formalism of spin precession with respect to a reference congruence
R. Jonsson – To be submitted to Class. Quant. Grav.

Paper VI
An intuitive derivation of spin precession in stationary spacetimes
R. Jonsson – To be submitted to Am.J.Phys.

Paper VII
Generalizing optical geometry
R. Jonsson, H. Westman – To be submitted to Class. Quant. Grav.

Paper VIII
Optical geometry across the horizon
R. Jonsson – To be submitted to Class. Quant. Grav.

Acknowledgements

First of all I would like to express my gratitude towards my supervisor, Marek Abramowicz for allowing me to work on what I find interesting – and to develop as an independent researcher. I would also like to thank Ulf Torkelsson and Sebastiano Sonego for all sorts of comments and advice.

I must of course also acknowledge the sole Bohmian-Mechanics-Lover of the house – who is also my friend, colleague and fellow rebel in physics – Hans Westman. We have shared more discussions and laughs (many at the expense of various physicists) than I can remember. 'Wha Wha Wha ...':)

While Achim Tassemark didn't include me in his acknowledgements, I am a forgiving man and would not in any way hold that against him:) He has been my companion in angling, longbow archery, smithery, horseback riding and much else over the years and likely will be in the future as well.

It can be debated whether they deserve it, but I would like to acknowledge also 'the most unlikely creatures of all' – the Lord of the String PhD-students below us (in every respect). May the spirit of Gert Jonnys guide you for ever:)

I would also like to mention the very best longbow shooting nephews of mine that the world has ever seen – David and Alexander:)

More important than anyone I must acknowledge Agneta – whose name is still magic, and who taught me that physics is just crap – compared to what matters in life.

Lastly I would like to acknowledge my parents, among many other things for making sure that I had something to eat while finishing this thesis. They are, as Tina Turner puts it 'Simply the Best' (although I very much doubt that Tina was referring to her parents:).

September, 2004

Rickard Jonsson

Grow stronger

From the 13'th warrior

Hope is for free

Fabrizio

Contents

1

Introduction

Ever since it was presented in 1916, Einstein's General theory of Relativity concerning space, time and gravitation, has been extremely successful in explaining all sorts of gravitational phenomena. Two examples from the beginning of the century are the gravitational deflection of light from distant stars passing close to our sun and the precession of the perihelion of Mercury. More recent experiments involve gravitational redshifts, relativistic slowing down of atomic clocks and indirect measurements of gravitational waves.

Since the theory was presented it has also puzzled the minds of physicists and people in general. Indeed, for most people the theory is still clouded in mystery. In particular, people find the legendary black holes fascinating yet incomprehensible.

To understand Einstein's theory one must understand its heart – the curved spacetime. Unfortunately it is easier said than done to explain this concept without using mathematics. There are however ways of working around the difficulties and two of my scientific papers are directly related to this.

In fact the whole process of explaining relativity has become something of a passion for me. I have spent countless hours doing computer-generated illustrations and more time than I care to remember down in the workshop making real models of curved spacetime. Using these models and illustrations I have given over thirty lectures on General Relativity in university courses as well as at high schools, science festivals and conferences.

I believe there is plenty of room for improvement in general of how we teach physics. The author of a book, whether popular or scientific, may have a crystal clear understanding of the field he writes about. As a reader translates the written words into his own mental language, a lot may be lost however. We can minimize this loss by using examples, working on the exact formulations and rewriting the text according to how well it was understood. Still, much of the intuition that we have concerning physics is encoded as mental images and movies. I believe the whole learning process can become much more efficient if we actually create these images and movies. Of course it takes a lot of effort to produce interesting and pedagogical illustrations. With the aid of better computers and software – our opportunities to do so have however improved. For fields like General Relativity, for which there is even a large interest from the general public, it might be worth the effort.

I hope to one day write a book on General Relativity directed towards a general audience. The idea is to use large color images spanning over double pages, to use no mathematics (or maybe just a little, well hidden in an appendix :) and to use analogies. I believe that one can give a very good understanding of General Relativity in this manner for anyone who is interested in the mysteries of Einstein's theories. The second chapter of this thesis gives an idea of what I want to achieve, although the layout and form of this thesis is limiting to say the least.

While all my work is related to General Relativity, it is not all related to popularizing the field. I have also worked on *inertial forces* and *gyroscope precession*. An example of an inertial force is the apparent force pushing us outwards as we take a steep curve at high speed with a car. A gyroscope is essentially a rapidly spinning body, that tends to keep its direction of spin. In general relativity we can however make the gyroscope turn or *precess* just by moving it around. For each of these two fields I have written both a formal paper and a paper more directed towards intuitive understanding.

I have also worked on the field of *optical geometry*. In brief, the idea behind this theory is to consider a rescaled (stretched) version of the standard spacetime. Relative to the rescaled spacetime some effects, like gyroscope precession, can be explained in a more straight-forward manner. In one paper I show, together with my colleague Hans Westman, how one can generalize the standard theory of optical geometry so it applies to a wider class of spacetimes than the standard theory of optical geometry does. In yet another paper I show how one may apply this generalization to consider optical geometry also for the inside of a black hole.

1.1 This thesis

This thesis serves many purposes apart from a being a means of collecting the appended papers. One purpose is to present my work, and the general field of work, for a more general audience. Another purpose is to present results that are not included in the papers. Lastly, the thesis provides an opportunity to review some topics that are underlying the appended papers. The thesis is organized in the following way:

- **Chapter 2** gives an introduction to General Relativity, aimed at a general audience. The main object of this chapter is to explain how spacetime *geometry* can explain why things are falling towards the Earth when we drop them.

- **Chapter 3** gives a brief introduction to the practical side of making illustrations using computer software like Matlab and PovRay.
- **Chapter 4** gives an account of how I made the real models that I use when lecturing on General Relativity.
- **Chapter 5** is a brief introduction to the topic of pedagogical techniques within general relativity, and my work therein.
- **Chapter 6** introduces the reader to inertial forces and my work within this field.
- **Chapter 7** is an introduction to gyroscope precession and my work related to this.
- **Chapter 8** gives an introduction to optical geometry and my two papers concerning this.
- **Chapter 9** provides some brief conclusions and an outlook.

Then follows a part of the thesis that contains technical comments on the papers. Two chapters are introducing new insights and results concerning Papers I & II, and two chapters are reviews.

- **Chapter 10** is a comment to Paper I, regarding the shape of the spacetime inside a star or a planet. I show that in the Newtonian limit it corresponds to an exact sphere, as depicted on the front page of this thesis.
- **Chapter 11** is a comment to Paper II where I present some additional results and insights that are not included in Paper II.
- **Chapter 12** reviews the kinematical invariants defined for a reference congruence of worldlines. This chapter is related to Papers III-VIII.
- **Chapter 13** reviews Lie transport and Lie differentiation. This chapter is mainly related to Paper III, although it also has some relevance for Papers IV-VIII.

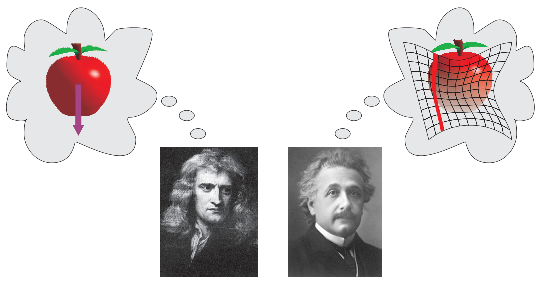
Then follow the appended papers.

2

An introduction to Einstein's gravity

Gravity is something that we are all more or less accustomed to. If we throw an apple upwards, or maybe of bowling ball, we know that it will soon fall down again.

One might wonder *why* everything that we throw upwards insists on falling down again. The two most renown theories for this were put forward by Newton (late 17'th century) and by Einstein (early 20'th century).



Newton's theory explains the fact that the apple returns by a gravitational *force* pulling the apple back towards Earth.

In Einstein's theory there is no gravitational force. Instead it explains the motion of the apple by a law saying that the *motion corresponds to a straight line in a curved spacetime*. To explain what this really means is the purpose of this chapter. Let us however warm up by explaining what a spacetime *diagram* is.

2.1 The spacetime diagram

Imagine that you are living on a straight staff with blue position markings on it (Fig. 2.1). On the staff a number of events happen: an alarm clock rings, a fire cracker explodes and a walking man puts down his feet.

Figure 2.1: **Life on a staff.**

We can mark the events happening on the staff in a spacetime diagram (Fig. 2.2). The later the event happens – the higher up in the diagram it is marked. If the event is far to the right along the staff, it is marked to the right in the diagram.

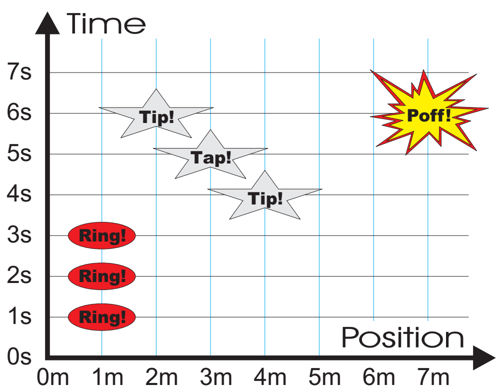


Figure 2.2: **The spacetime diagram**. *Here we can mark everything that happens on the staff. Can we deduce from the diagram in what direction the man is moving?*

In the diagram we can also illustrate *motion* of objects by so called *worldlines* (Fig. 2.3). The higher the velocity – the more tilted the worldline (if straight up is considered as not tilted).

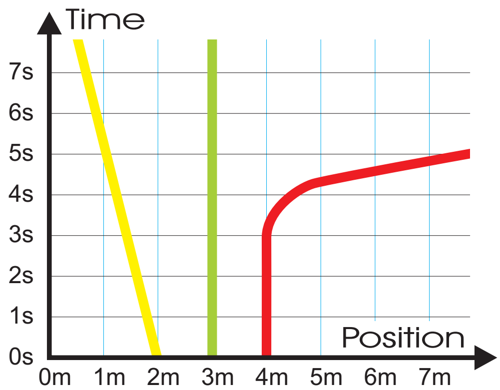


Figure 2.3: **Worldlines describing motion**. *The leftmost line corresponds to an object moving towards the left along the staff, the middle to an object at rest and the rightmost to an object that is first at rest but is then accelerated to high velocity towards the right.*

Soon we will discuss the concept of curved spacetime illustrated as a more or less curved surface. The flat spacetime diagram can then be regarded as a map of the curved spacetime, similar to how a map can illustrate the curved surface of Earth.

2.2 Flat spacetime

We now place the staff somewhere out in empty space, where we do not notice any effects of gravity (Fig. 2.4).

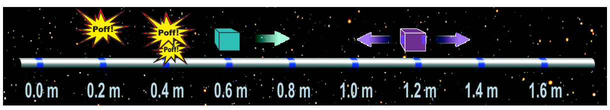


Figure 2.4: **A staff in space**. *Along the staff various events happen and objects are moving.*

In Einstein's theory we have here a so called *flat* spacetime which we can illustrate by a flat plane (Fig. 2.5). On the plane we describe events and motion along the staff in space, just like we did earlier in the spacetime diagram.

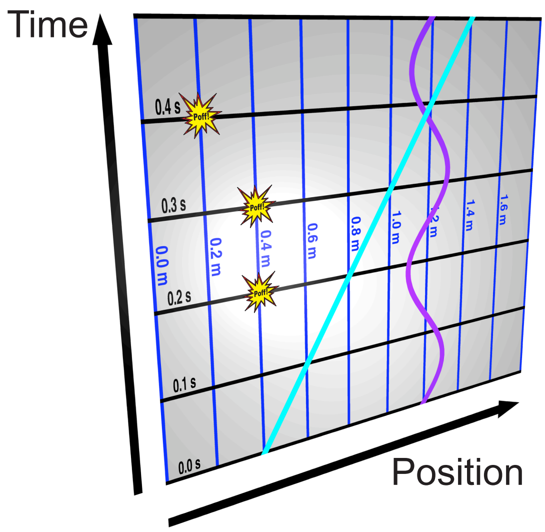


Figure 2.5: **A flat spacetime for a staff in space**. *The upright blue lines correspond to fix position along the staff. What type of motion does the two worldlines correspond to?*

If we throw an apple along the staff in space, and let the apple move freely, it will continue with *constant* velocity along the staff. In Einstein's theory the motion of the apple is explained by a law saying that the motion corresponds to a straight line in the spacetime.

> The motion of a thrown free object corresponds to a ***straight*** line in the spacetime

By *free* we here essentially mean that nothing is touching the object. In section 2.4 we will give it a more precise meaning.

To create a straight line on our flat spacetime surface, we could use a ruler. We will however instead use a little toy car equipped with a downwards directed pen (see the cover of this thesis). The toy car, from now on denoted the *drawing car*, we can then by hand roll forward. The point of using the drawing car is that it also works on curved surfaces, as will be used later.

If we know the initial velocity of a thrown apple, we can use the drawing car to *predict* how the apple is going to move. We put the drawing car at a point on the spacetime surface corresponding to the starting position of the apple and the time that we threw the apple. We give the drawing car a *direction* relative to the surface, corresponding to the starting velocity of the apple, and roll the car forward. The drawn line corresponds to the motion of the apple after the throw.

As an example we may predict the motion of two apples, thrown with different velocities from the zero-meter position at zero time (Fig. 2.6). Can we see were the apples are predicted to be after 0.4 seconds?

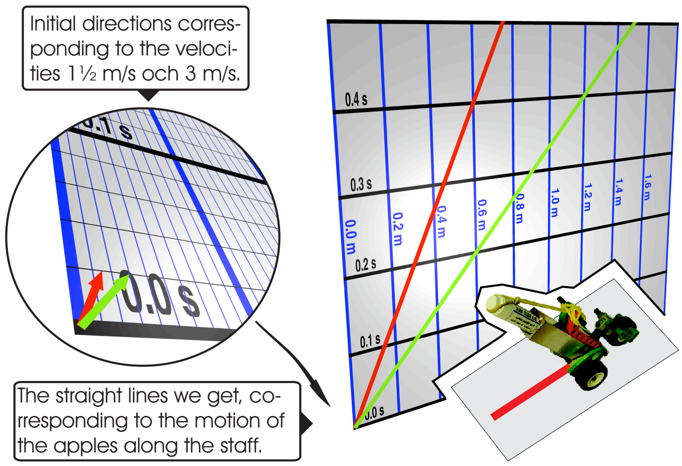


Figure 2.6: **Straight lines in a flat spacetime.** *The lines can be created with a three-wheeled toy car - equipped with a pen.*

Fig. 2.7 illustrates how the scenario with the apples would look in reality.

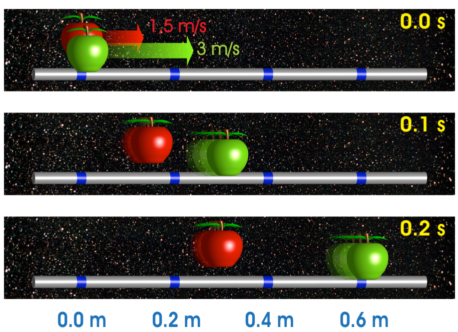


Figure 2.7: **Apples thrown along a staff in space.** *The images show the position of the apples at consecutive times. Compare with Fig. 2.6.*

We see how one, at least in outer space, can predict the motion of thrown apples using a geometric model.

2.3 Mass curves spacetime

We can create a curved spacetime from the flat spacetime we just showed by stretching and bending it. In Einstein's theory spacetime is curved by *mass*. The greater the mass the greater the curvature (Fig. 2.8).

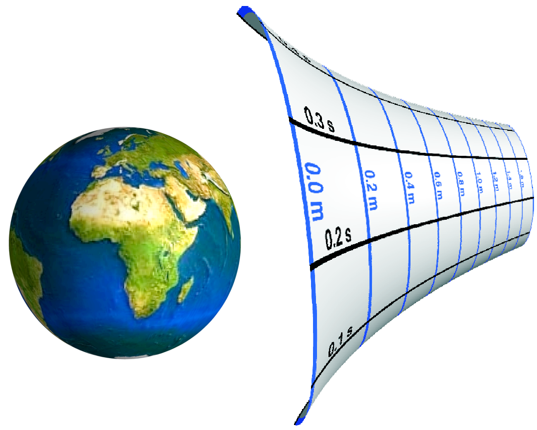


Figure 2.8: **Mass curves spacetime.** *The Earth illustrates in what direction the curving mass lies. Strictly speaking the mass should lie* **within** *the spacetime – not outside as depicted here.*

Sometimes one illustrates how mass curves something by putting a metal ball on a rubber sheet and look at how the sheet bends. Unfortunately this analogy tells nothing about how the *spacetime* is curved which is what we want to illustrate here. Before we talk more about how mass curves the spacetime we will explain what it means that it *is* curved.

2.4 Curved spacetime for a staff outside Earth

We now place the staff upright outside of Earth. Here the mass of the Earth has curved the spacetime as illustrated by the funnel-shaped surface of Fig. 2.9.

Time is directed clockwise around the funnel (as seen from above) and one circumference corresponds to one second. An object moving up along the staff at a fixed speed corresponds to a worldline spiraling around the funnel. The faster the object moves, the steeper the spiral. An object at rest corresponds to a horizontal circle.

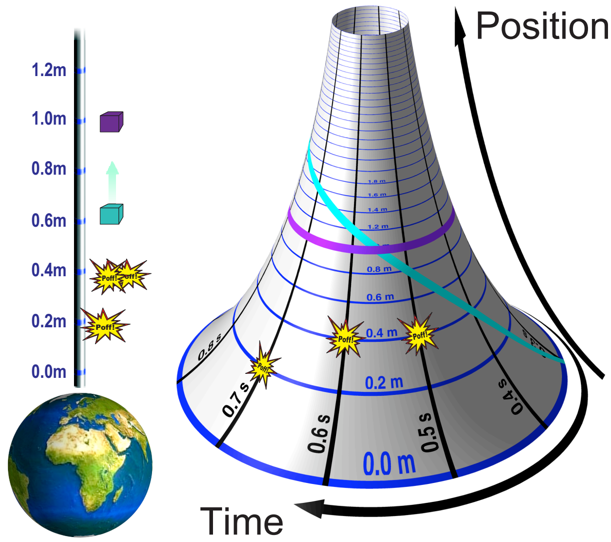


Figure 2.9: **A curved spacetime for a line outside Earth.** *Strictly speaking the surface should not close in on itself in the time direction. Rather one should come to a new layer after one circumference – as on a paper roll, illustrated on the back cover of this thesis. The Earth is shown in miniature.*

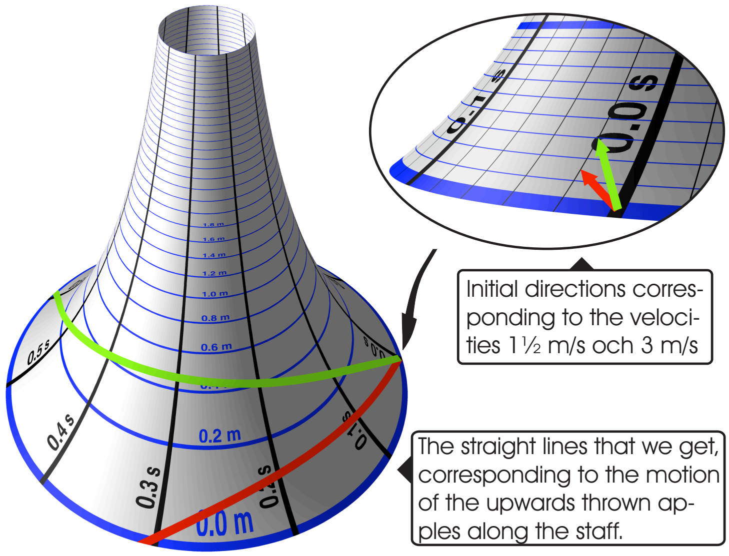


Figure 2.11: **Straight lines in a curved spacetime.** *Note that the light-colored (green) initial direction corresponds to a precisely twice as high velocity as the dark-colored (red) initial direction. Time is directed clockwise as seen from above.*

In Einstein's theory, here as well as in outer space, the motion of thrown *free* objects corresponds to straight lines in the spacetime. By free we mean that gravity alone determines the motion of the objects (no air resistance for instance). We can thus predict the motion of the thrown objects by the same method that we used earlier (with the drawing car), although the spacetime surface is now curved.

As an example, we study two apples thrown upwards from the zero-meter marking on the staff with different velocities (Fig. 2.6).

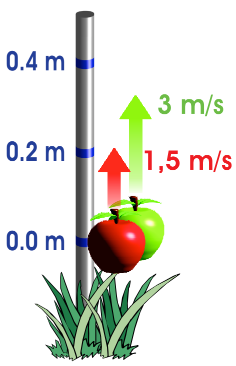


Figure 2.10: **Apple-throwing.** *As soon as the apples have left the throwing hand, the motion corresponds to straight lines in the spacetime.*

We thus place the drawing car on the base of the funnel, corresponding to the starting position on the staff. We give it initial directions relative to the funnel, corresponding to the initial velocities of the apples, and roll it forward along the surface of the funnel (Fig. 2.11).

We see that both worldlines correspond to apples that initially are moving up along the staff, reach a maximum height, and then fall back again. The light-colored (green) line reaches a higher level on the funnel and takes a longer time before returning to the base of the funnel. This is reasonable since the corresponding apple is thrown upwards with a higher velocity. You may note precisely how high the apples are expected to come, and what time it takes them to return to their initial position. In Fig. 2.12 we illustrate the motion of the apples in reality.

We thus have a geometrical model for predicting the motion of apples. Unlike in Newton's theory there is no gravitational force in this model. It is the shape of spacetime that determines that the worldline returns to the base of the funnel, just like an upwards thrown apple returns to the surface of the Earth.

If the spacetime outside of the Earth would

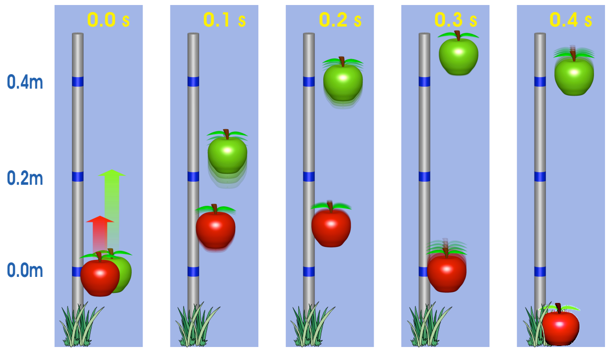


Figure 2.12: **Upwards thrown apples along a staff outside of Earth.** *The pictures show the position of the apples at consecutive moments in time – compare with the predictions from Fig. 2.11.*

have a shape as illustrated to the left in Fig. 2.13 – apples thrown upwards would not return according to the theory. Note also that if we let the drawing car *turn* – it will not predict the motion of upwards thrown free objects (illustrated to the right in Fig. 2.13.

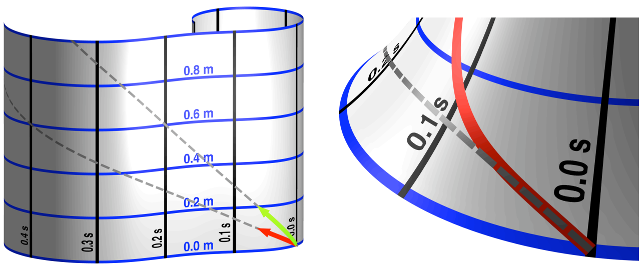


Figure 2.13: **Left:** *An alternative shape of spacetime. A drawing car rolled straight forward would follow the dashed lines that never return to the base of the spacetime.* **Right:** *An alternative worldline. A drawing car that turns created the solid line which is not returning to the base of the spacetime.*

Thus the shape of the spacetime *and* the rule about straight lines determine that the upwards thrown apples should return to Earth – in Einstein's theory.

In the introduction we mentioned that we would attempt to explain the meaning of a *straight line in a curved spacetime*, and how this can explain why an upwards thrown apple falls down again. Hopefully this feels rather natural now.

2.5 The spacetime for a line through the Earth

To give another example of a curved spacetime we imagine a hole straight through the Earth, with a staff that follows the hole (Fig. 2.14).

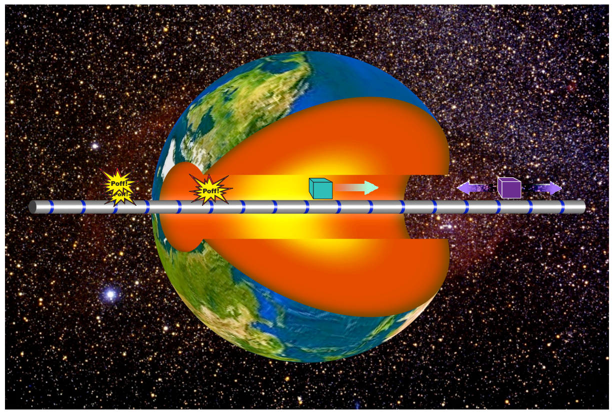


Figure 2.14: **A staff through Earth**. *Along the staff events happen and objects move.*

The curved spacetime for the staff is illustrated in Fig. 2.15. Notice that the outer parts of this spacetime correspond to the funnel that we earlier displayed.

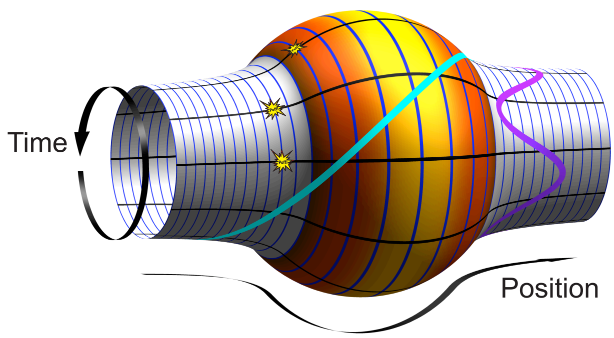


Figure 2.15: **Curved spacetime for a staff through Earth**. *In this illustration it is about 8 minutes between the time lines (running along the surface) and about 1500 km between the position lines (going around the surface).*

We now move to the center of the Earth where we simultaneously throw three apples, with different velocities, along the staff as illustrated in Fig. 2.16.

This time it is perhaps not as evident what is going to happen. But we know that as soon

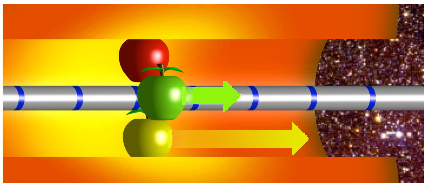


Figure 2.16: **Apples in the center of the Earth**. *The dark-colored apple is just released and the others are thrown with the velocities 6 and 18 km/s.*

as the apples have left the throwing hand their motion will correspond to straight lines in the spacetime.

If we have an actual model of this type of spacetime we can repeat the procedure with the drawing car. We put the drawing car on the spacetime model at a point corresponding to the center of the Earth and a certain time. We give the drawing car three different initial directions corresponding to the three different velocities. The result is shown in Fig. 2.17.

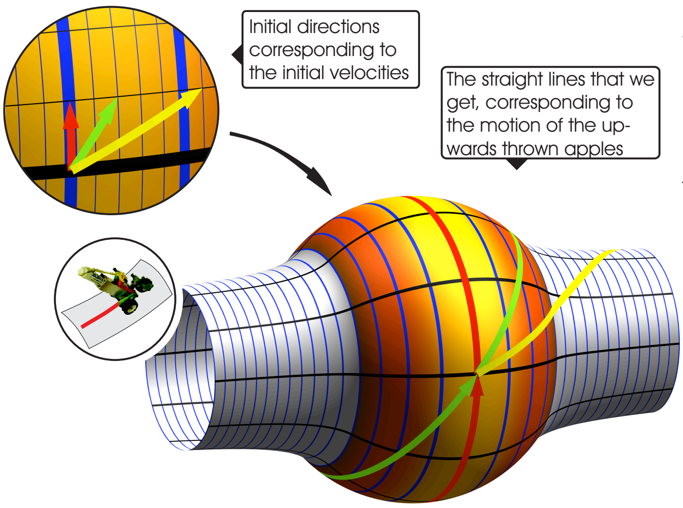


Figure 2.17: **Straight lines in a curved spacetime.** *Note that the straight lines correspond to motion along the staff through the Earth.*

The dark (red) line corresponds to an apple at rest at the center of the Earth, the semi-light (green) line corresponds to an apple that is oscillating back and forth along the staff, around the center of the Earth. The light (yellow) line corresponds to an apple that passes the surface of the Earth and continues onward into outer space without ever returning.

These scenarios correspond precisely to what we would expect from reality (although an actual experiment would be difficult to carry out in practice). Once more we see how straight lines in a curved spacetime can explain the motion of thrown apples.

As we move further away from the Earth the funnels at the ends of the spacetime will assume a cylindric shape. But cylinders are in fact *flat* in the sense that we can roll out a cylindric surface to a flat surface. When we are far from the curving effects of the Earth we thus have a flat spacetime, just like in the preceeding section.

In this chapter we have illustrated different parts of the spacetime in a certain order. All of these parts are however connected and there is thus only *one* spacetime (per universe:).

2.6 Forces and gravity

In Einstein's theory there is no *gravitational force*, but there are *forces* also in this theory. An example is the force by which we affect a car as we push it forward. A force acting on an object causes the worldline of the object to *curve*. The greater the force the greater the curvature of the worldline.

> The motion of an object affected by a force corresponds to a ***curved*** line in the spacetime

As an example we study an apple that at first is just hovering near a staff in outer space. The apple then receives a push towards the right and goes off along the staff at a constant speed. How this scenario looks relative to the flat spacetime is illustrated in Fig. 2.18.

The law about forces and the curvature of worldlines applies also when the *spacetime* is curved. As an example we study an apple that we hold by its shaft here at Earth. In Einstein's theory there is *only* an upwards directed force from the hand acting on the apple.

How the apple can remain at rest even though

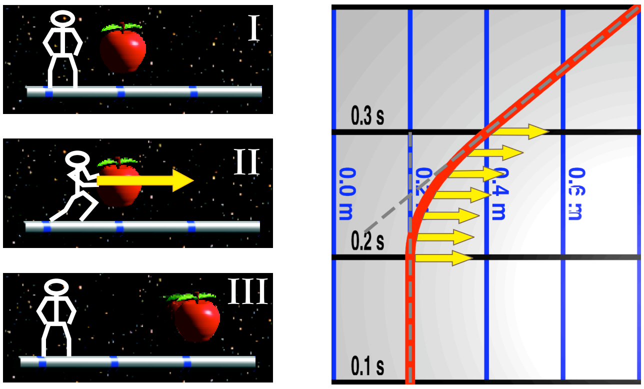


Figure 2.18: **Apple-pushing in outer space**. *The worldline deviates from the straight dashed lines as the force of the push acts on the apple.*

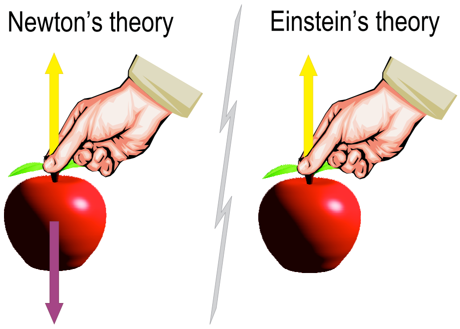


Figure 2.19: **An apple held at rest at Earth**. *In Newton's theory there is a gravitational force acting on the apple. In Einstein's theory there is no gravitational force.*

it is affected by a net force upwards may appear pardoxical – but the solution is given by the curved spacetime (Fig. 2.20).

If we were to direct the drawing car along a horizontal position line and roll it forward, it would have to turn *upwards* to remain at the same height (the front wheel should be turned to the right). The apple is thus at all times affected by an upwards force, just like the drawing car is all the time turning in the upwards direction – but it does not get upwards because the spacetime is curved!

To make the apple go upwards we must act on it by a greater force than that required to keep it at rest. The drawing car must thus turn more upwards than what is required to keep it at a fix height. The result would be a worldline spiraling up along the funnel.

If we on the other hand were to drop the apple, such that no forces are acting on it, the worldline would straighten up (Fig. 2.21).

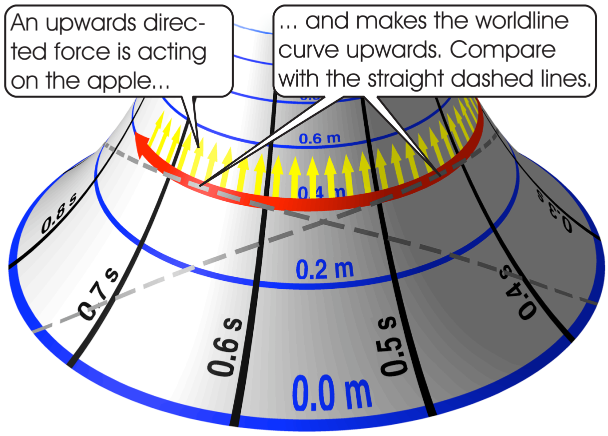


Figure 2.20: **The resolution of the paradox**. *An apple at rest at a height of 0.4 meters. The force makes the worldline curve upwards all the time but the shape of the spacetime insures that it will not get any higher anyway.*

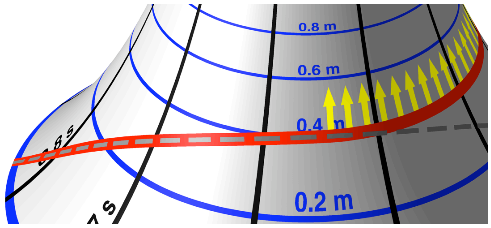


Figure 2.21: **An apple being released**. *As soon as the apple is released the worldline follows a straight line.*

Note that there is no gravitational force pulling the apple down after we have released it. The apple falls because we cease to curve its worldline and let it follow a straight line.

2.7 About mass

The effect of forces in Einstein's theory is to curve worldlines. How much a worldline is to curve for a certain force depends on the *mass* of the object. The greater the mass the lesser the curvature of the worldline (Fig. 2.22). One might say that a large mass makes the drawing car hard to steer – so it tends to roll straight ahead unless a great force acts on the object.

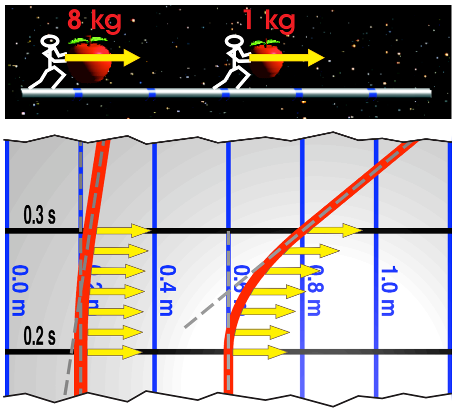


Figure 2.22: **Apple pushing in space – again**. *The worldline of the heavy apple is curved less by an equal force.*

That it is difficult to affect the motion of an object with a large mass is common to both Einstein's and Newton's theory. The difference lies in that large mass does not mean large gravitational force in Einstein's theory – there is no force of gravity there.

2.8 More about mass

Apart from the effect that mass has on the curvature of worldlines, it has also the property that it curves *spacetime* itself. The point is that mass contains energy, and all sorts of energy curve the spacetime. Even a radio signal emitted by a cellular phone contains energy and will curve the spacetime a little.

So it is not quite as simple as that an upwards thrown apple corresponds to a straight line in a spacetime whose shape is independent on the apples motion, but rather its a straight line in a spacetime that the apple itself partially curves (or has curved). In the case of the apple, the mass is however so small that it it does not curve the spacetime significantly. If we on the other hand would let for instance the *moon* fall towards the Earth (and the Earth towards the moon), we would have to take into consideration that the Earth and the moon consist of particles that all curve the spacetime a little. The spacetime would also for this case correspond to an unmoving surface (it would make no sense to have a moving spacetime), but the shape would be more complicated than those we have shown in this article. It would not be rotation symmetric. On the irregularly shaped surface we would however still be able to use the drawing car to predict the motion of an apple thrown from the Earth towards the moon.

2.9 Was Newton wrong and was Einstein right?

We have seen, although in a simplified form (that nevertheless gives precisely the right predictions), how Einstein's theory explains the motion of apples by a curved spacetime. Newton on the other hand explained it with a *gravitational force*.

One might then wonder who was right? Both theories describe with good accuracy all sorts of every day gravitation, like how a dropped apple falls towards the ground.

There are however situations where the predictions from Newton's theory differ from those of Einstein's. A famous example regards the orbit of Mercury around the sun (Fig. 2.23).

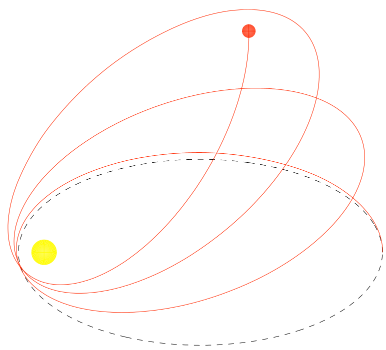


Figure 2.23: **The orbit of Mercury**. *The dashed line is the motion as prescribed by Newton's theory, the solid by Einstein's. The effect that the ellipse is rotating (in the plane of the paper) is however exaggerated half a million times.*

Measurements of the position of Mercury from the 18'th century and onwards show that the almost elliptical orbit of Mercury is in fact rotating in accordance to Einstein's theory.

So Newton's theory does not describe reality correctly. It is however simple to use for calculations compared to Einstein's theory, and works very well in most cases. But does this mean that Einstein's theory about the curved spacetime is right?

When it comes to physics at a fundamental level, one can never *know* if a theory is right. Tomorrow the apples may fall upwards and we see no way to explain this in Einstein's theory. Then we must search for a new theory that explains both why the apples fell down yesterday and why they are falling up today.

How nature *really* works – we do not know. What we can say for sure is that the curved spacetime exists in Einstein's *theory* – which seems to be a good description of nature.

Figure 2.24: **Curved spacetime in practice**. *Demonstration using a spacetime funnel and drawing car.*

3

Making illustrations

The images of this thesis have mainly been created using Matlab, under the Linux operating system. Post processing has been made in Corel Photo Paint. Images built from several different images or vector graphics have been made in Corel Draw, gimp or xfig.

There are considerably more advanced programs for making three-dimensional graphics than matlab. The ideal would likely be 'Maya' – the program that was used to make the animated movie 'Shrek' among others. Learning some basic tricks and building your own specialized graphic functions in matlab and utilizing a freeware called povray, one can still come a long way.

3.1 About matlab

To create the images of this article I have written over 170 matlab functions and programs of a total of more than 350 kb. To list them all would fill this entire thesis. Just to give a feel for how the various codes are working I give a little example of a matlab code utilizing some of my functions on the next page.

The non-standard matlab functions that I call here are `rjxsurf`, `putonsurf` and `rjsurfset`. The function `rjxsurf` plots the surface much like the standard matlab function `surf` does. The main difference lies in that all the lines of the vertexes of the surface need not be plotted but for instance one may plot every other line by including `'linesep1',2` in the arguments. To use this feature one must also send the viewing angles `az` and `el`. What `rjxsurf` does is that it puts the lines out as ordinary lines (using `plot3`), but then it lifts them a bit towards the eye – so they are visible on top of the surface. This function allows for the creation of smooth surfaces that are not completely riddled with too dense-lying lines. Also setting `lighting phong` smooths the surface nicely.

```
clf
N=50;
[X,Y,Z]=sphere(N);
C=ones(size(X));
az=160;el=20;

%%%% Plot the sphere with coordinate lines %%%%%%%%%%

rjxsurf(X,Y,Z,C,'setlines','on','az',az,'el',el,'linesep1',2,...
'linesep2',2,'col',[0 0 1],'linecol',[1 1 0]);

%%%% Set out a 'surface line' on the sphere %%%%%%%%%%

phibase=linspace(0,2*pi,N+1); % A coordinate basevector
thetabase=linspace(0,pi,N+1); % A coordinate basevector
phi=linspace(2.8,5.2,200); % Trajectory phi
theta=pi/2+sin(phi*10).*(phi-min(phi))*0.2; % Trajectory theta

rjcell{1}={'surfline',theta,phi,'col',[1 0 1],'epsilon',0.1,...
'arrow','curved'};
handles=putonsurf(X,Y,Z,thetabase,phibase,{az,el,[]},rjcell)
rjsurfset(handles,{0.7,0.3,0.4})

%%%% Fixing some lighting etc %%%%%%%%%%

view(az,el);
axis off;axis image
shading flat
light;lighting phong
```

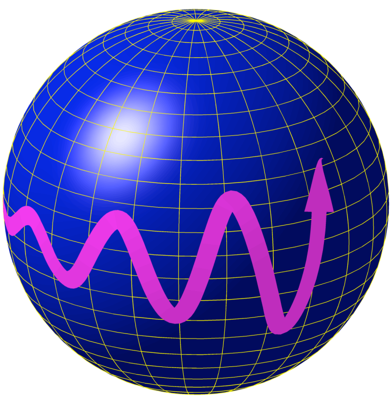


Figure 3.1: The output from the matlab program above

Printing to a file at high resolution and resampling (compressing) the image while applying anti-aliasing can give high quality images. The function `putonsurf` is especially well suited for plotting worldlines on parameterized surfaces. The function `rjsurfset` is just a quick way to set some reflection properties of surface elements.

3.1.1 Some shortcomings of matlab

Matlab does most of what one wants concerning graphics, but it lacks some features that would be useful:

- Doing real raytracing, i.e allowing objects to cast shadows and reflections on other objects. In Matlab there is only a cruder shading of surfaces depending on the angle of the surface relative to the light source - but independent on the reflections and shadows of other surfaces.
- Doing anti-aliasing while rendering. 'Rendering' is the process whereby a two-dimensional image is formed from a three-dimensional scenario. Anti-aliasing is a certain averaging technique of the color of nearby pixels, taking away the 'staircase' appearance (pixelization) of the edges of lines and surfaces and making them smooth.
- Doing high resolution animation. This is perhaps the most surprising deficit of matlab. It has a simple structure for making animations (movies). There is however no real way to control the resolution of the movie – matlab appears to take a so called screen-shot of every image. In general the screen resolution (and size) are not what one might desire for a reasonably nice (anti-aliased) movie that covers the screen of a standard computer or TV.
- Doing subtractions of volumes. For instance one may want to subtract a cylinder from a sphere to make a hole through the sphere. This type of operation one must do 'by hand' in matlab – and it can be quite time consuming and complicated.

Concerning the second point there is a solution. Matlab can print to a file at a very high resolution, for instance issuing the command:

```
print -dtiff -r900 filename
```

The dimensions of this image is 8×6 inches. At 900 dots per inch this is a very large image (typically several tens of Mb). Then one opens the resulting file `filename.tif` in some program for manipulating two-dimensional images (Corel Photo paint in my case). Then one resamples the image to 600 dpi and to the size that one desires, making sure that the anti-alias check-box is checked. The required resolution of the original file to get a good final result depends on the dimensions of the final file. As a rule of thumb one should have at least twice as many pixels (per dimension) in the original image as one does in the final image, to get a nice anti-aliasing.

3.2 About povray

There are graphics programs that have the abovementioned features that matlab lacks. In particular there is a program called PovRay, a freeware downloadable from **http://www.povray.org**. I advice the interested reader to go there and have a look at the 'Hall of Fame' images. Some are really spectacular in appearance, and still comparably easy to make.

PovRay works similar to Matlab, but it is specifically designed to make images rather than to do calculations. Here is an example of a piece of povray code in a file called `example.pov`:

```
#include "colors.inc"
background {color Cyan}
camera {
location <0, 2, -3>
look_ at <0, 1, 2>}

sphere {<0, 1, 3>, 1
texture {pigment{color Red}
finish {reflection {0.5} phong 0.7 phong_size 10 ambient 0.0 diffuse
0.8}}}

plane { <0, 1, 0>, -1
pigment {checker color Green, color Blue}}

light_source { <2, 4, -0.001> color White parallel point_at
<0,0,0>}
```

To process (render) this file one may write (on the Unix or Linux system):

```
povray +P +A +W1200 +H900 example.pov
```

The image is then rendered and the resulting file is output as `example.png`. If one desires another format, like the postscript file displayed below in Fig. 3.2, one opens `example.png` in some program (like gimp or Corel Photo Paint) and converts it to the desired format.

There are however also downsides to povray compared to Matlab. For instance one cannot interactively rotate and zoom the three-dimensional scenario using the

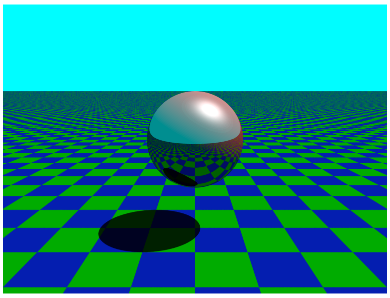


Figure 3.2: An example of the output from a short povray program.

mouse. Also the matlab standard representation of surfaces as matrixes (which is not standard in povray) is quite practical if one is dealing with parameterized surfaces, as I have been to a large degree.

3.3 Matlab to povray

I have written a converter (a matlab script) that creates a povray file from a matlab generated three dimensional scene . The converter is called `rjmat2pov.m` and is very simple to use. After running your matlab script to create a three dimensional image, type `rjmat2pov('filename')` and you will get a povray file called `filename.pov` that you may render as any other povray file. What `rjmat2pov` does is that it finds all the visible surface elements in the matlab scene, and divides them into the natural triangles that they are made of and makes a povray `mesh2` object out of them. The colors and reflection properties are also translated although the translation of the reflection properties is rather approximate (different types of reflection properties are used). Any light sources are also translated from the matlab scene to the povray scene. There is plenty of room for improvement of this code. Unless there are very many surface objects and large colormaps it should work just fine however. I used this converter for the cover illustration.

3.4 Comments

I plan to put the matlab files required for the examples above on my homepage. There are quite a few help-files required, but so long as you put the whole pack in your personal matlab directory (assuming Linux or Unix) so that matlab can find them – this need not concern you. My current homepage is at **http://fy.chalmers.se/~rico** and the files can be found by clicking the matlab symbol.

4

Making models

A picture may be worth more than a thousand words, but a model is worth more than a thousand images (or at least more than a couple of images:).

If for no other reason than that it was a whole lot of work to make them, and that I would like to be the first to write a thesis on relativity containing phrases like ’turning-lathe’, ’welding’ and ’molding form’, I will now briefly explain how I made the models that I use while lecturing about gravity.

4.1 The funnel shaped spacetime

From aluminum sheet metal (5 mm thick) I cut out 10 equal stripes whose shape I had calculated and printed (using a computer). I then waltzed these stripes to give them the right curvature in the direction along the funnel. Next I made several press-forms of different radii and pressed the the stripes in the press-forms to give them the right curvature in the direction around the funnel. Then I TIG-welded together the pieces, see Fig. 4.1. Next I built a rack consisting of an axle and several wooden discs that fitted snuggly to the inside of the funnel. Using the rack I turned the spacetime round and smooth as best as possible with the lathe cutting steel. (I am skipping the parts where I cut through the surface and had to weld it together again:) Then, while spinning it fast in the lathe, I used an angle grinder with a disc of segmented grinding papers to further smoothen the surface. After that followed finer grinding and polishing in the lathe. In the end it shone like a mirror.

I then left it to a car lacquerer, Mr Istvan Papp, who lacquered it with a grey metallic and clear varnish (two component).

Next I put the funnel into the lathe, spun it slowly, and used a water based felt tip pen in stead of a cutting steel to make the blue circles around the spacetime. I cut the tip of the pen (and spent quite a few pens) to make the lines of varying width

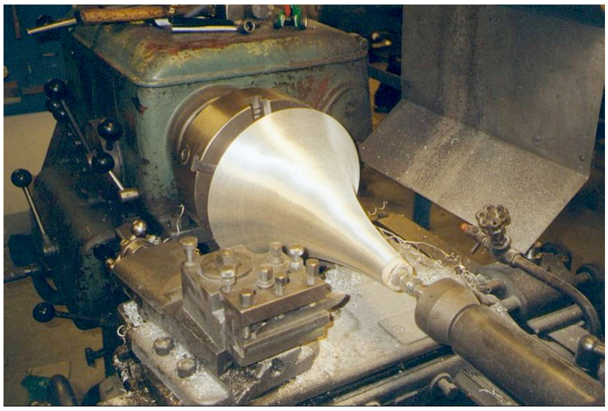


Figure 4.1: A spacetime in the making. To the left the spacetime has just been spot-welded together. To the right the spacetime after turn lathing and angle grinding

(narrower towards the top of the funnel).

For the black time lines along the funnel, I used masking tape and paintbrush. For the second and meter markers I used sticker letters, the part that remains after you have taken out the actual letter, and used a paint brush to make the letters and numbers.

Between applying lines and text one had to put a layer of clear varnish to protect what was already made. I am still indebted to Istvan for helping me out with this. At last I lacquered the inside myself with a blue metallic and clear varnished it.

4.2 The bulgy cylinder spacetime

For the second model I used a different technique. First I glued together boards of wood to form a big lump, with an metal axle running through the middle of it. Then I turned and ground this big chunk of wood to a shape corresponding to half the spacetime, see Fig. 4.2.

To make it as smooth as possible I clear varnished, waxed and polished the wooden plug. Then I applied gelcoat (a plastic), layers of glass fibre and liquid plastic to make a glass fibre molding form. In this form I molded the two ends of the spacetime, again using glass fibre and liquid plastic. If you want to try this – do not neglect the useful-

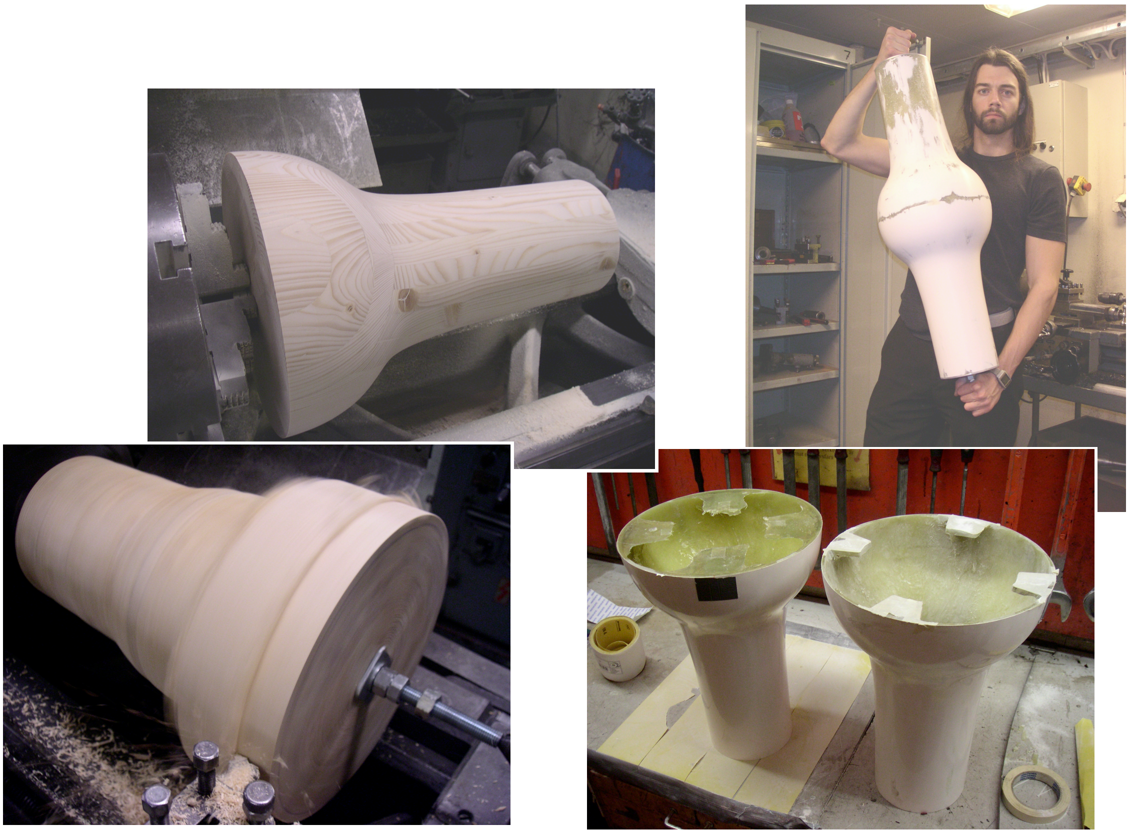


Figure 4.2: From a lump of wood to a glass fibre spacetime.

ness of a really good gas mask. After cutting and grinding the ends of the spacetime flat, I molded a means of attachment with bolts between the two ends. I also did some molding at the ends of spacetime allowing me to tightly fit a couple of turned aluminum discs there. Through central holes in these discs I put a steel axle enabling me to put the hole construction in the lathe and to grind away the small differences in radius at the intersection.

Then came the *Time of Lacquering* - that I did myself this time. Below is just the briefest outline of a few things that went wrong in the primary tries.

- `Metallic paint too old.`
- `Air holes in glass fibre becoming visible during lacquering.`
- `Partially clogged up jets on the paint sprayer.`
- `Wrong distance from sprayer to target.`
- `Dust particles on the surface getting trapped under the paint.`
- `Dust particles in the air sticking to the drying surface.`
- `Paint running from a jet getting thrown out on the surface.`
- `Water drops falling on un-dried paint.`

- `Trying to remove paint from a failed lacquering attempt - the thinner eats through the lower layers of paint creating a canyon in the surface.`
- `The fire department arriving, clearing the entire house, because the lacquering fog set of the alarm.`
- `The color appearing patchy upon rotating the surface - although it looked fine when not rotating it.`

Then I built a scaffolding for the entire spacetime allowing me to rotate it. Seeing that there was an un-roundness of something like a millimeter, I decided to put the whole thing in the lathe again. After a lot of spray puttying and grinding it was back to the lacquering box. Then of course all the air holes in the glass fibre surfaced and I had some nice sessions of applying spray putty and grinding it in the lathe again. Then it was back to the lacquering box. To avoid patches becoming visible when rotating the spacetime, I hooked it up to an electric drill rotating the spacetime as I just moved the paint sprayer from one end to the other. A few more things went wrong.

- `Putting a finger in the un-dried paint to see if the paint is not dry yet -- a classic!`
- `The spacetime spinning to fast - paint drying to fast.`
- `Un-smoothness of the surface from the last lathing becoming visible as the metallic is applied.`

In the end I made more than 15 lacquering attempts – and every time you have to wait for the paint to dry, grind the spacetime, flush the lacquer box and yourself (wearing a rain coat) etc. I even left it to a professional car lacquerer at one point, but I was not happy with what he had done either, so I ground it again and kept going. Suffice it to say that in the end I made it.

Oh – just in case you want to try this – some inspectors found out about my technique with the electric drill and they were not overly happy to have a spark-inducing machine inside a flammable fog of thinner:)

Putting the coordinate lines on the surface was analogue to what I did on the funnel. Parallel to making the bulgy spacetime, I also made a flat spacetime model, using a plane of PVC-plastic. My models are displayed in Fig. 4.3.

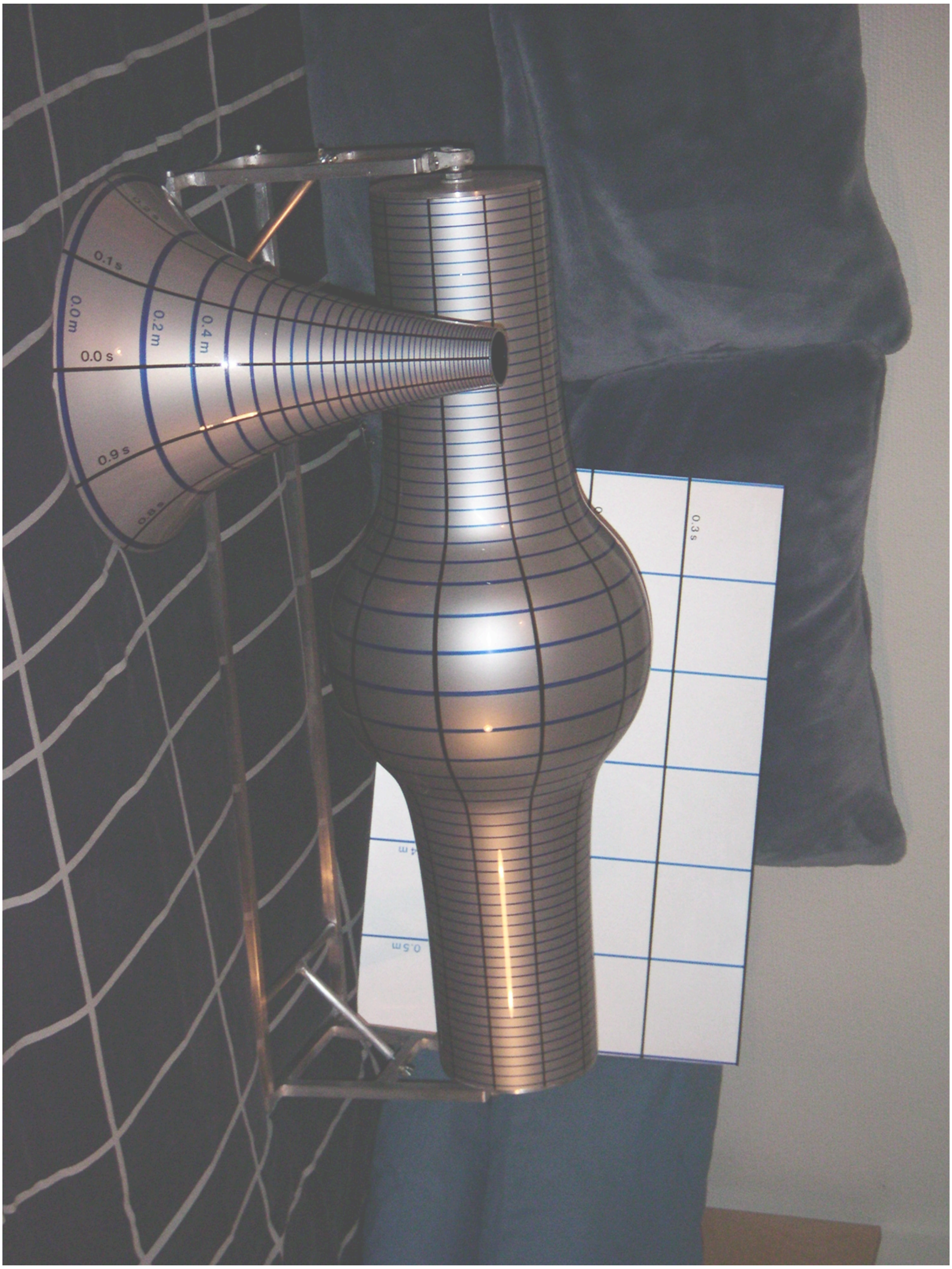


Figure 4.3: Some curved shapes on my bed:)

5

Spacetime visualization

I have written two research papers where I develop two different techniques directed towards popularizing the field of general relativity. Both these techniques allows us to visualize curved spacetime by a curved surface. The ideas underlying the illustrations are however quite different, and they illustrate different aspects of the full theory of General Relativity.

Similar techniques have been developed before. There is a popular scientific book called 'Visualizing Relativity', by L. C. Epstein [1]. Some of the illustrations are quite similar to the ones that I have created (related to Paper I) although the underlying idea and how one interprets the illustrations is completely different[1].

Yet another technique is presented by D. Marolf in [2] though interpreting the embedded surface that he considers requires some knowledge of special relativity [2].

There is also a paper by W. Rindler [3] where the basic ideas have some similarity to those of Paper I though he considers a more mathematical approach intended for undergraduate students of physics. [3].

---

[1]The theory underlying [1] is based on the assumption of an original time-independent, diagonal, Lorentzian line element. Rearranging terms in this line element one can get something that looks like a new line element, but where the proper time is now a coordinate. The 'space-propertime' can be embedded as a curved surface.

[2]In particular he considers the radial line element of a maximally extended black hole. The proper distances can be illustrated by embedding the surface in 2+1 dimensional Minkowski space (visualized as a Euclidean 3-space)

[3]Essentially he considers the Newtonian equations of motion in Hamilton's formalism, and matches them (approximately) with the geodesic equations of motion we get considering a four-dimensional Riemannian (positive definite) line element. The components of the metric are found using qualitative guessing and trial and error. Thus he presents a way of bypassing special relativity to anyway give an understanding of the concept of curved spacetime – given some knowledge of Newtonian mechanics and Riemannian geometry.

5.1 Paper I

In this paper I present a method that allows us to visualize curved spacetime by a curved surface. The method is tailor-made to explain what it means to have *straight lines in a curved spacetime* and how this can explain why an apple thrown upwards returns to Earth. This method is underlying the introduction to general relativity presented in chapter 2, as well as the cover of this thesis.

The idea underlying this method is hard to explain in brief without getting technical, but I will try to give a feeling for what is done in the paper.

Consider two events, like snapping your fingers, first with the right hand and then with the left. In Einstein's theory, one assigns an *interval* (a kind of distance) between this pair of events. If there is time for a light signal to travel from one event to the other, the interval will be *positive* otherwise it will be *negative* [4].

Next consider the curved surface of a sphere on which we have sprinkled grains of sand. The geometry of the sphere can be defined as the set of distances separating all pairs of (nearby) grains of sand on the surface. Similarly, the geometry of spacetime can be defined as the set of intervals separating all pairs of (nearby) events. Relative to this abstract geometry, a canon ball shot from a canon will follow a line that is *straight* in some (abstract) sense. The fact that we have negative intervals (distances) between events however makes it impossible to illustrate directly the spacetime geometry by a sphere or some other curved surface.

What I do in Paper I is that I take the set of spacetime intervals between events, and transform them such that all the intervals become positive, but without changing what is a straight line. We can illustrate the new geometry, that has only positive distances, with a curved surface.

5.1.1 A technical note

In technical terms, I consider a Lorentzian two-dimensional, time-independent line element. I then find another line element that is positive definite but has the same geodesic structure. The new geometry can be embedded in Euclidean space. Due to a one-parameter freedom in going from the original to the new *dual* line element, together with freedoms of the embedding, one can illustrate even the weakly curved spacetime of Earth with a significantly curved surface.

5.2 Paper II

In this paper I present another method of visualizing curved spacetime. This method is well suited to explain why clocks slow down near massive bodies, and what that

[4]Strictly speaking it is the square of the interval that can be positive or negative – but that is of little importance here.

really means. The method can also be applied to explain black holes, expanding universes (big bang) and much more. This is a technical paper exploring the possibilities of the method rather than applying the method. However, in section 2 of the paper the method is applied to give a very brief introduction to general relativity. The section is directed to teachers of physics, and uses only a little bit of mathematics, though in principle the method can be used without any reference to mathematics. Even at the current level, a general reader with an interest to learn about curved spacetime may benefit from reading it. Fig. 5.1 gives an example of an illustration from Paper II.

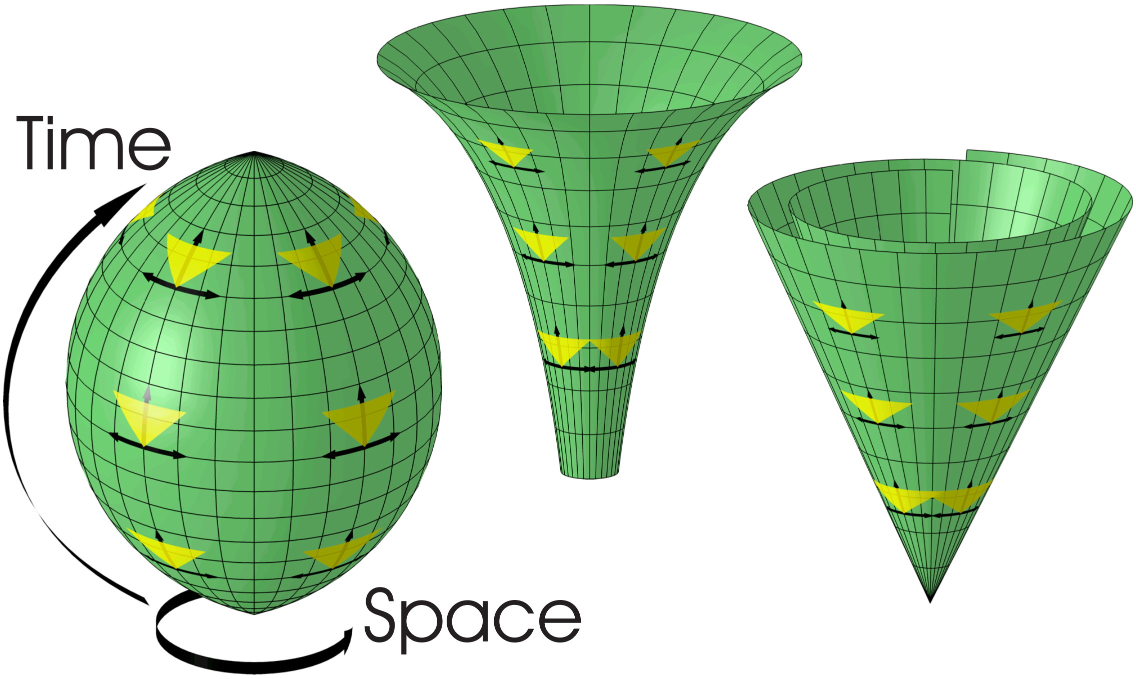


Figure 5.1: Cosmological models. From these models one may understand how space itself can expand. For further details, see section 2 of Paper II.

As was the case in the previous paper, the idea is to make a transformation turning all of the intervals (distances) between pairs of events into *positive* distances. This time the idea is however not to preserve straight lines, but to preserve the intervals themselves (as much as possible). The resulting geometry can be illustrated with a curved surface. If one knows how to interpret such an illustration, one will be able to find out everything there is to know about the true spacetime geometry (including the negative distances).

5.2.1 A technical note

Technically speaking, I first introduce an arbitrary field of timelike four-velocities u^μ. Then, at every point, I perform a coordinate transformation to a local Minkowski system comoving with the given four-velocity. In the local system, the sign of the spatial part of the metric is flipped to create a new metric of Euclidean signature which for the special case of two dimensions be embedded as a curved surface. On the surface lives small Minkowski systems relative to which special relativity holds.

6

Inertial forces

An example of an inertial force is the (apparent) force that pushes us outwards if we are on a rotating platform, or if we drive our car through a roundabout at high speed. This particular inertial force is known as a centrifugal force.

In the general theory of relativity one can introduce inertial forces in a similar manner to how one introduces them in Newtonian mechanics. Some effects that occur in relativity are however quite counter-intuitive from the point of view of Newtonian mechanics. In fact if we consider a rocket in orbit near a black hole it will require a *greater* rocket thrust outwards to keep it from falling into the black hole the faster it rotates around the black hole[1]. We might say that the centrifugal force is pointing inwards rather than outwards here.

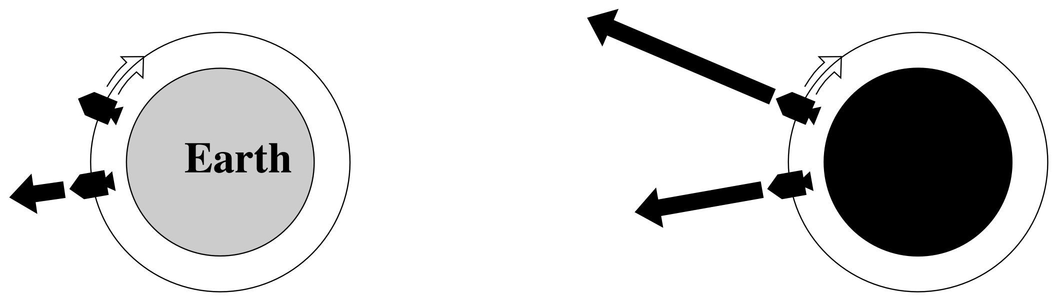


Figure 6.1: **Left:** a rocket orbiting the Earth. The faster the orbital velocity the less outward thrust from the rocket engine is required to keep the rocket on a fix radius. If the orbital velocity is high enough – no outward thrust is required. **Right:** a rocket in orbit around a black hole. The faster the rocket moves the *greater* the required rocket thrust needed to keep it from falling into the black hole.

[1]This effect occurs between the event horizon (the radius of the black hole) and 1.5 times the radius of the black hole (the so called photon radius where particles of light can move on circles).

I consider inertial forces in general spacetimes using the mathematical formalism of general relativity in Paper III. I also derive a general formalism of inertial forces using only basic principles of relativity as presented in paper IV. The interested reader who knows a bit about special relativity and vectors may benefit from reading this paper. Among other things, I show how one can explain the abovementioned scenario with the black hole as a natural consequence of relativity. The mathematical results of the formal paper are a bit more general than those of the intuitive paper, but the results agree where comparable.

6.1 A technical note

In Paper III we consider a general timelike congruence of reference worldlines in an arbitrary spacetime. The local motion of a test particle can be described in terms of the velocity relative to the congruence, the time derivative of this velocity and the spatial curvature of the test particle worldline projected onto the local slice. In this formalism inertial forces appear naturally in form of the kinematical invariants of the congruence (see chapter 12). While relativistically correct, the resulting equations of motion are effectively three-dimensional. I show that when the congruence is shearing, the projected curvature is not necessarily the most natural measure of spatial curvature, and I present an alternative definition. I also study the effect of conformal rescalings on the inertial force formalism.

In Paper IV, via the equivalence principle and basic elements of special relativity such as time dilation, I derive the same formalism using only three-vectors for the special case of a non-shearing congruence.

7

Gyroscope precession

A gyroscope is basically a symmetric body that spins very fast around an axis and is suspended in such a way that there is no torque acting on it. If we move the gyroscope along a circle it will keep pointing in the same direction according to the theory of Newtonian mechanics. According to special relativity however, if we transport the gyroscope very fast along the circle – its spin axis will turn, or in other words *precess*. As we go to general relativity the situation becomes even more interesting. For instance, if we move the gyroscope along a certain circle around a black hole – the gyroscope spin axis will (automatically) turn in such a manner that the spin axis is always directed along the direction of motion (see Fig. 7.1).

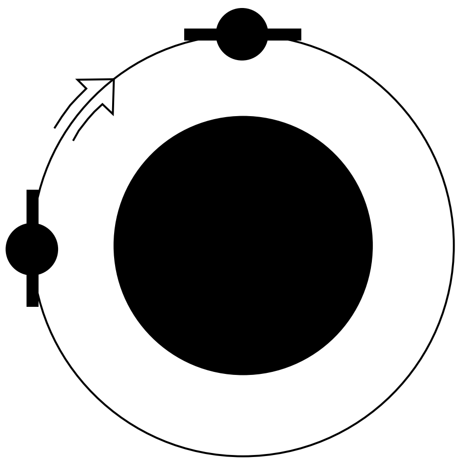


Figure 7.1: A gyroscope moving on a circle around a black hole. The spin axis is the thick bar through the sphere. Despite the fact that we are not affecting the gyroscope by any torque, it still turns.

I consider gyroscope precession in general spacetimes in Paper V using the mathematical formalism of general relativity. I also consider gyroscope precession in a very non-formal manner in Paper VI. Here I use basic principles of relativity and intuition rather than formal mathematics. In particular I show how one can use the special relativistic explanation to explain also the general relativistic effects. The formal paper is a bit more general than the intuitive paper but the results match where comparable.

7.1 A technical note

As was the case in the inertial force analysis, the idea rests on introducing an arbitrary congruence of timelike worldlines and expressing the spin precession with respect to this frame of reference. Rather than considering the standard spin vector S^{μ}, I consider the spin vector we would get if we would *stop* the gyroscope by a pure boost relative to the congruence. The stopped spin vector obeys simple laws of rotation and is ideally suited for this approach. In Paper V I use a four-covariant formalism starting from the Fermi-Walker transport equation, and derive an effectively three-dimensional formalism of spin precession.

In the intuitive paper I arrive at the same result, considering a non-shearing reference frame (congruence) using only three-vector formalism together with the equivalence principle and special relativity. I do not use the Fermi Walker transport equation in this paper. In particular I show that one may regard the gyroscope precession in arbitrary stationary spacetimes as a double Thomas precession effect. One part comes from the gyroscope acceleration and the other from the reference frame acceleration (there is also a trivial contribution from any rotation of the reference frame).

8

Optical geometry

The mass of a star curves the fabric of space and time. How space is curved is illustrated in Fig. 8.1.

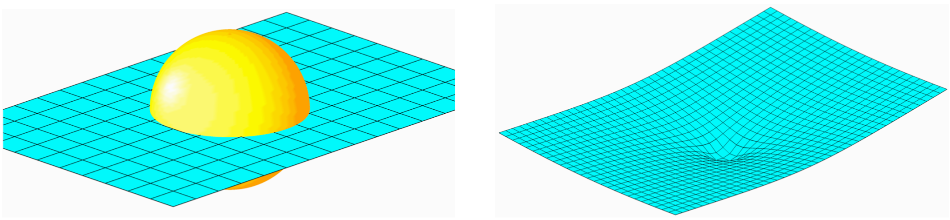


Figure 8.1: Illustrating the curved geometry of a plane through a star.

While the velocity of light is everywhere locally the same, clocks near the star will run slow relative to clocks far from the star (for some understanding of this, see section 2 of Paper II). Effectively this means that as we send a light signal from one end of the star to the other (imagine that the star is transparent), it will take a *longer* time for the light signal than we might think considering only the spatial distance that the light needs to travel. Seen from the outside, there is an apparent slowing down of the light signal. We can account for this by considering a rescaled space, where the extra distance accounts for why the light takes such a long time to reach the other side of the star. This stretched space is known as the *optical geometry*, see Fig. 8.2. Relative to the optical geometry, photons (particles of light) move along straight spatial lines. This is not generally the case for photons. If we send out a photon horizontally – it will fall away from a straight line just like anything else we might throw in a horizontal direction. The only difference is that photons move so fast that we do not see them fall. But relative to the optical geometry – they do follow straight lines. In [16] M. Abramowicz gives a popular scientific presentation of optical geometry.

Together with my collaborator Hans Westman, I present in paper VII a way of

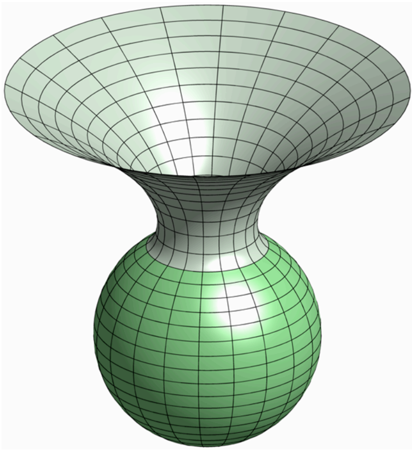


Figure 8.2: The optical geometry of a plane through a star. It is related to the standard geometry of the plane by a stretching. The darker region lies within the star and the lowest point is the center of the star. Unlike the curved surfaces presented prior to this chapter, this is a visualization of curved space and not of curved spacetime.

generalizing the standard theory of optical geometry to include arbitrary spacetimes (so that the optical geometry may change in time). In paper VIII I consider specific applications of the generalized optical geometry. In particular I consider a black hole, including its interior.

8.1 Generalized optical geometry in technical terms

In an arbitrary spacetime we may introduce a spacelike foliation specified by a single function $t(x^\mu)$. Forming the covariant derivative of this function we get a vector field orthogonal to the foliation. We then introduce a congruence of worldlines parallel to the vector field. Performing a conformal transformation we rescale away time dilation, so that in coordinates adapted to the slices and the congruence we have $\tilde{g}_{tt} = 1$ (here the tilde indicates a rescaled object). The new spatial geometry that we get is the optical geometry. Relative to this geometry a photon moves a unit distance per unit coordinate time. Assuming the congruence to be shearless we show that a a geodesic photon will follow a spatially straight line, and a gyroscope following a straight spatial line will not precess relative to the direction of motion. Also the sideways force required to keep a test particle moving on a straight spatial line will be independent on the velocity. Using a novel measure of spatial curvature introduced in Paper III we show that also for the case of a shearing congruence, photons follow straight lines and the sideways force keeping a test particle moving along a straight line is independent on the velocity. In paper VIII I consider a non-shearing congruence radially falling towards a black hole, allowing us to consider the extended optical geometry across the horizon.

9

Conclusion and outlook

Concerning my work with inertial forces, gyroscope precession and optical geometry – I have largely accomplished what I set out to do. At the moment, I see no particular issues here that I would like to further resolve.

Concerning the spacetime visualization techniques, I am more or less content concerning the theoretical background. I am however a little curious to find out what can be done in the dual scheme considering time dependent line elements. Also I am bit curious about for instance how the spacetime of a radial line through a collapsing shell of matter would look like in the absolute scheme. Mainly however, my interest concerning the pedagogical techniques lies in the visualization as such. I have plenty of ideas of how one can go beyond the illustrations that I have shown in this thesis to further inspire and teach the physics community and the general community about the marvels of curved spacetime. As I mentioned in the introduction, I hope to use my ideas and methods to write a book about relativity for a general audience.

As for my interest in physics I tend towards the foundations of physics. How *can* one unite general relativity with quantum mechanics? Maybe if we crack this nut we will get a grip on the measurement problem of quantum mechanics, and perhaps even consciousness itself. Or maybe it will be the other way around. In any case, I suspect that a revolution in the way we think about the universe will be necessary to accomplish this fusion. As soon as I have finished this thesis – I will get started on it. It shouldn't take me very long to crack that nut:)

Comments on the Research Papers

10

A spherical interior dual metric

This chapter is a comment on Paper I, concerning the shape of spacetime for a radial line inside a planet of constant proper density. It presumes knowledge of the notation of Paper I.

We know that in Newtonian theory, a particle that is in free fall around the center of a spherical object of constant density, will oscillate with a frequency that is independent of the amplitude. This would fit well with an internal dual space that is *spherical*. We saw in Paper I the possibility to choose parameters β and k that produces substantial curvature also in the case of the weak gravity outside our Earth. It is natural to ask whether we really can find parameters such that the internal geometry becomes spherical. In the following sections we will see that this is not generally the case but one can choose parameters such that it is exactly spherical in the weak field limit.

10.1 Conditions for spheres

For a sphere of radius R, we introduce definitions of r and z according to Fig. 10.1. Using the Pythagorean theorem it follows that:

$$\frac{dr}{dz} = -\sqrt{\left(\frac{R}{r}\right)^2 - 1} \tag{10.1}$$

We let $z = z(x)$, where (see Paper I, Eq. (23)):

$$\frac{dz}{dx} = \sqrt{c(x) - r'^2} \qquad \text{here} \qquad r' = \frac{dr}{dx} \qquad c = g_{xx} \tag{10.2}$$

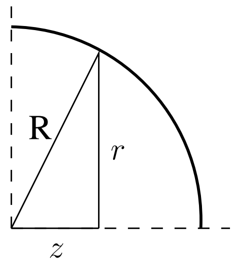


Figure 10.1: Definitions of variables for a sphere. The axis of rotation (around which time is directed) is horizontal.

We readily find:

$$R^2 = \frac{r^2}{1 - r'^2/c} \tag{10.3}$$

Also if we have a rolled-up sphere (see Fig. 6 of Paper I) such that $r = \gamma r_{\text{sphere}}$ and $r' = \gamma r'_{\text{sphere}}$ this is modified to:

$$R^2 = \frac{r^2}{\gamma^2 - r'^2/c} \tag{10.4}$$

10.2 The dual interior metric

For the Schwarzschild interior dual metric we may show that [1] :

$$r'^2/c = \frac{\beta k^2 x^2}{4x_0^6(a_0 - \beta)} \qquad \text{and} \qquad r^2 = \alpha k^2 \frac{a_0}{a_0 - \beta} \tag{10.5}$$

Inserting this into Eq. (10.4) we readily find that for the embedding to correspond to a sphere of radius R and rolling parameter γ we must have, for *all* $x < x_0$:

$$R^2\gamma^2(a_0 - \beta) - R^2\frac{\beta k^2 x^2}{4x_0^6} - \alpha k^2 a_0 = 0 \tag{10.6}$$

Notice that if we divide the expression by αk^2 we will see that any change in R^2 and γ can be canceled by a corresponding change in α and k respectively.

For Eq. (10.6) to be true for all x it must be true for every power in x. We note that a_0 has a lot of powers in x which would mean that the factors multiplying a_0 would have to vanish. But then the x^2-term in the middle cannot be canceled by anything. Thus the expression *cannot* be true for all values of x and thus the interior dual metric is not exactly spherical.

[1] The first relation can most rapidly be obtained re-juggling Eq. (32) in Paper I (replace $<$ with $=$ and remove the "min"), look also at Eq. (23) for understanding. The second is the square of Eq. (21) in Paper I.

10.3 Approximative internal sphere

We *can* however produce something that is very similar to a sphere. Set $\gamma = 1$, corresponding to a non-rolled sphere, and $\alpha = (1-\beta)/k^2$ meaning unit radius at infinity. We can expand the two occurrences of a_0 in Eq. (10.6) to second order in x^2. Demanding the equation to hold to zeroth, first[2] and second order yields after some simplification (Mathematica does fine simplifications):

$$\beta = \frac{a_{00} \cdot (R^2-1)}{R^2 - a_{00}} \qquad \text{where} \qquad a_{00} \equiv a_0(0) = \frac{1}{4}\left(3\sqrt{1-\frac{1}{x_0}} - 1\right)^2 \tag{10.7}$$

$$k = \frac{2x_0^{7/4}}{\sqrt{3\sqrt{x_0-1} - \sqrt{x_0}}} \tag{10.8}$$

Notice that Eq. (10.8), coming directly from the second order demand, is independent of β. The first relation is nothing but Eq. (21) in Paper I, where one has demanded $r = R$ and $\alpha = (1-\beta)/k^2$, taken at $x = 0$.

Inserting β and k from Eq. (10.7) and Eq. (10.8) into a plotting program yields pictures with a very spherical appearance. The way it works is that the center of the bulge has the exact radius and curvature of a sphere, then the rest is not exactly spherical. In the Newtonian limit however, where $x_0 \to \infty$, we *do* get a perfect sphere. This is true whichever finite radius R we choose, as is explained below.

10.4 Spheres in the Newtonian limit

From Eq. (10.3) we have the necessary relation for a sphere:

$$\frac{r'^2}{c} = 1 - \frac{r^2}{R^2} \tag{10.9}$$

Using the specific expressions for β and k given by Eq. (10.7) and Eq. (10.8) we may evaluate both the left and the right hand side of Eq. (10.9) to lowest non-zero order [3] in $1/x_0$, to find that the equality holds *exactly* for all x to this order. Thus in this limit we get an exact sphere. In fact we knew in advance that the equality would hold. The reason is that, a_0 expanded to second order in x for which the equality holds, is the same (or in fact a bit more exact) as a_0 expanded to first order in $1/x_0$ [4]. So, in the Newtonian limit the interior dual metric is isometric to a sphere for any value of β (so long as R remains finite).

[2] It is always satisfied to first order

[3] Terms like x^2/x_0^3 are of course treated as $1/x_0$-terms

[4] If we instead would have had $a_0 = 1 - \sqrt{1 - x^3/x_0^4}$ which expanded in x to second order is zero, while *not* being zero expanded to first order in $1/x_0$ (remember that x^3/x_0^4 is of order $1/x_0$), we could not have used this little trick.

An embedding of the Earth spacetime is displayed in Fig. 10.2. Here I have chosen the radius of the central bulge to be twice the embedding radius at infinity. The parameters β and k are given by Eq. (10.7) and Eq. (10.8) with $x_0 = 7.19 \cdot 10^8$, suitable for the Earth. As can be seen the bulge is quite spherical.

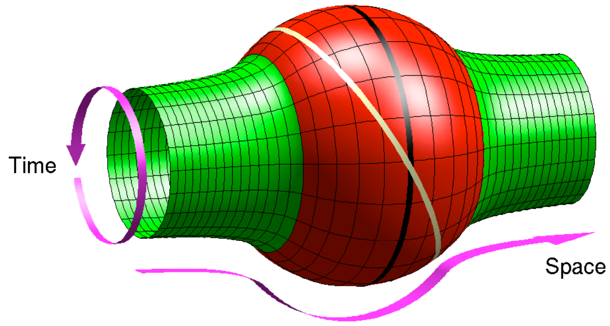


Figure 10.2: An embedding of the dual spacetime of a central line through the Earth. The time per circumference is roughly 84 minutes.

Notice the worldlines of the two freefallers on the interior sphere – one static in the center – the other oscillating around the Earth with a period time of roughly 84 minutes.

11

The absolute visualization

In this chapter I take the opportunity to comment on some issues concerning the *absolute* approach to spacetime visualization (Paper II). I will consider only two-dimensional applications.

I start by giving some intuitive understanding for how changing generators affects the visualization. I then move on to find what the vacuum field equations look like in 2 dimensions assuming a flat absolute geometry (so the freedom lies in the twist of the generators).

I will also consider some toy-models of theoretical interest, in particular illustrating spacetimes with timelike loops. I will also give an example of how one may choose generators such that outgoing photons (for instance) follow absolute geodesics for the line element of a radial line through a black hole.

11.1 The perch skin intuition

In special relativity one can look at the active Lorentz transformation as a shift of the physical spacetime points along hyperbolas, as illustrated to the left of Fig 15 in Paper II. Alternatively we may view the Lorentz transformation as a two-step stretch and compress process along the light cone coordinates u and v. Stretching by a certain factor along u means compressing by the same factor along v (so areas are preserved) as depicted in Fig. 11.1.

This process of stretching and compressing is directly related to the absolute visualization scheme. If we choose generators in the leftmost rhomboidal section of spacetime in Fig. 11.1, such that the generators are rotated a certain angle clockwise from the upwards direction – then the absolute geometry would be that of the surface to the right (think of the generators in the comoving coordinates). Hence changing the generator corresponds to a compression and a stretch along the local lightcone

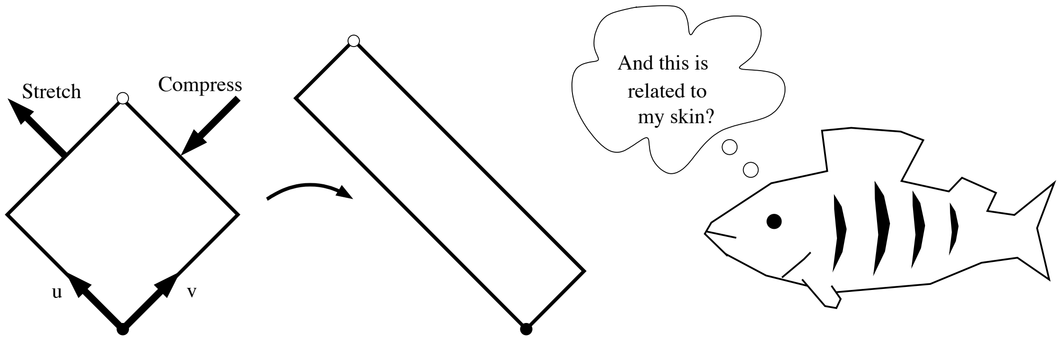


Figure 11.1: An active Lorentz transformation corresponds to a stretch and a compression.

coordinates. This reminds me of how a perch skin is behaving. Pulling in the skins length-direction makes the skin automatically shrink in stripe direction [1].

We realize that for any embedded surface in the absolute visualization technique, we can *locally* stretch and compress it to give a different appearance to the same physical spacetime. In other words we can do any deformation that preserves local areas and keeps the null lines at right angles. As an application of this newfound intuition, we consider a deformation of a Lorentzian flat spacetime illustrated by a flat absolute metric with uniformly directed generators – to a new flat absolute metric where the generators are curving, see Fig. 11.2.

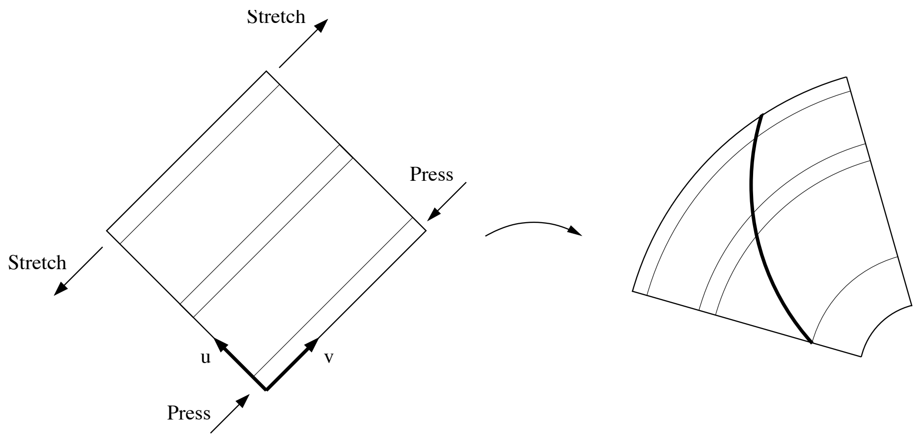


Figure 11.2: A deformation of flat spacetime. All the lines are null except the thick curve which is a generating worldline. Any deformation that preserves the angles between null lines and preserves local areas will leave the Lorentzian spacetime unaffected.

[1]As soon as my father starts hooking all those perches that he claims are biting his lure, I can start making more extensive tests of the properties of perch skin:)

11.2 Vacuum field equations for flat absolute metric

In the absolute visualization scheme we have have a direct visual means of finding the proper distances between nearby events. Maybe we could also build some intuition of how to find out whether there is a proper curvature or not. For this purpose, we study a simple case of flat absolute geometry where the Lorentzian geometry is determined by the direction of the generators, specified by an angle α as depicted in Fig. 11.3.

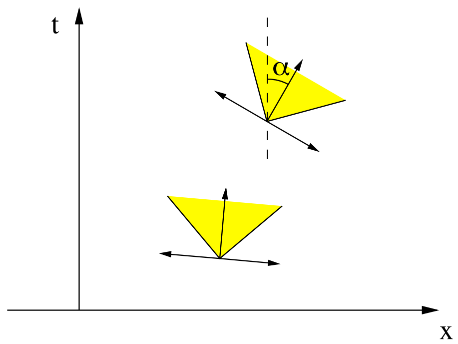


Figure 11.3: The direction of the generators (the local Minkowski systems) specified by an angle α.

The angle $\alpha(t, x)$ goes clockwise from the t-axis on the plane to the local t-axis of the local Minkowski system. The absolute four-velocity of the generators has the form:

$$u^\mu = (\cos(\alpha), \sin(\alpha)) \tag{11.1}$$

We find the Lorentzian metric through $g_{\mu\nu} = -\delta_{\mu\nu} + 2u_\mu u_\nu$ as:

$$\bar{g}_{\mu\nu} : \begin{bmatrix} -1 + 2\cos^2(\alpha) & 2\cos(\alpha)\sin(\alpha) \\ 2\cos(\alpha)\sin(\alpha) & -1 + 2\sin^2(\alpha) \end{bmatrix} \tag{11.2}$$

We may calculate the Ricci tensor for this using **grtensor** in Mathematica, but the expressions become quite large. To simplify matters, we choose our t-axis to coincide with the local Minkowski time axis. Then the angle α is small in a region around this point and we may Taylor expand the metric (to second order in α):

$$\bar{g}_{\mu\nu} : \begin{bmatrix} 1 - 2\alpha^2 & 2\alpha \\ 2\alpha & -1 + 2\alpha^2 \end{bmatrix} \tag{11.3}$$

Calculating $R_{\mu\nu} = 0$ and setting $\alpha = 0$ with **grtensor** yields a single equation:

$$(\partial_x\alpha)^2 - (\partial_t\alpha)^2 + \partial_x\partial_t\alpha = 0 \tag{11.4}$$

So here we have the vacuum field equations [2] , assuming a flat absolute geometry. Notice that Eq. (11.4) holds in a point, where the t-axis is chosen to coincide with the generator. Notice also that the two first terms are first order derivatives whereas the last is a second order derivative. At any point we can thus choose an arbitrary derivative in the local t and x-direction and still have a locally flat spacetime – so long as $\partial_x \partial_t \alpha$ is given by Eq. (11.4).

11.3 On geodesics and flat metrics

As an application of the field equations above, consider the special case of a flat absolute metric in 2D, and geodesic generators (i.e straight lines). Notice that the generators do not necessarily have to be parallel, but could for instance extend from a point and outwards (like spokes on an old cartwheel), or something more complicated.

Assuming the generators to correspond to straight lines, we have everywhere:

$$\nabla \alpha \cdot \hat{n} = 0 \qquad \text{where} \qquad \hat{n} = (\cos(\alpha), \sin(\alpha)) \tag{11.5}$$

Here α is the angle describing the tilt of the local generator as depicted in Fig. 11.3. In Cartesian coordinates (t, x) where the t-axis is parallel to the generators at the point in question, we have $\alpha = 0$ and thus Eq. (11.5) yields $\partial_t \alpha = 0$. Differentiating Eq. (11.5) with respect to x, and then setting $\alpha = 0$ yields:

$$(\partial_x \alpha)^2 + \partial_x \partial_t \alpha = 0 \tag{11.6}$$

Comparing with Eq. (11.4) we see that the vacuum field equations are satisfied. Thus for geodesic generators on the flat plane the corresponding Lorentz geometry is flat.

For an arbitrary embedded spacetime in the absolute visualization scheme – consider turning the lightcone (the local generator) by $90°$ everywhere while letting the shape of the surface remain. This would exactly correspond to changing the sign of the Lorentz metric. Changing the sign of the metric does not affect the affine connection, and thus not the curvature either. It then follows that also if we have straight lines *orthogonal* to the generators for a flat absolute geometry – we have a flat Lorentzian geometry [3].

Indeed looking at the rightmost figure in Fig 15 of Paper II we have a good example of a flat absolute metric (embedded as a cone) with straight generators in some parts, and straight orthogonal lines in another part, that is Lorentzian flat.

[2] In two dimensions the vacuum field equations imply flat Lorentzian spacetime.

[3] We could also show this directly assuming that we everywhere have $\nabla\alpha \cdot \left(\cos(\alpha + \frac{\pi}{2}), (\sin(\alpha + \frac{\pi}{2})\right) = 0$. It follows analogously to the above derivation (differentiate with respect to x this time) that also for this case the vacuum equations are automatically satisfied, hence spacetime is flat.

11.4 Closed dimensions and timelike loops

The absolute illustration method is well suited for making spacetime toy models. Such illustrations can be valuable for thought experiments and qualitative reasoning. In this section we illustrate a few spacetimes that are closed in one or two dimensions. In particular, images as that depicted to the left in Fig. 11.5 are useful to illustrate how spacetime can be something much more complex than just space plus time.

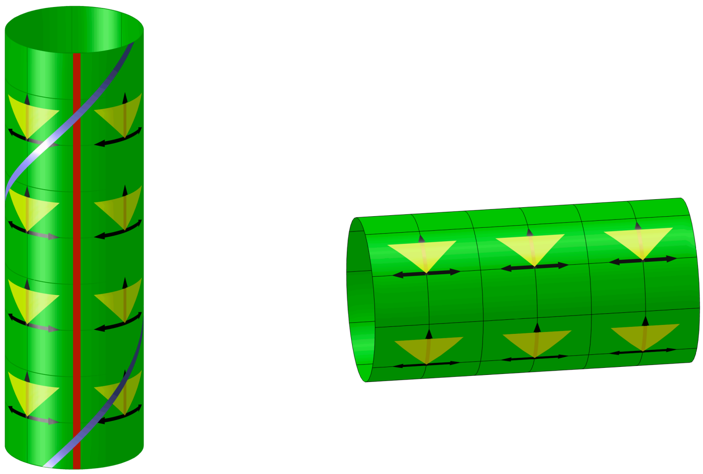


Figure 11.4: To the left a spacetime that is closed in space, with the trajectories of two inertial observers. To the right a spacetime that is closed in time.

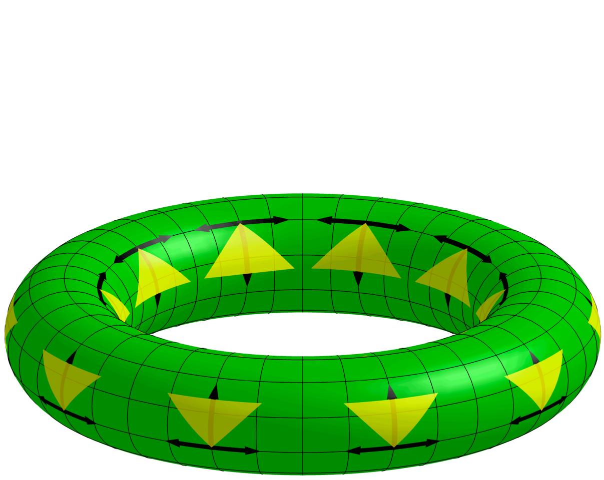


Figure 11.5: To the left a spacetime that is closed in space in some regions (top and bottom) and closed in time in the middle region. To the right a spacetime that is closed in space and in time.

An observer whose worldline winds as depicted to the left in Fig. 11.5 would as young see himself as old and vice versa. The thought that easily comes to mind is – 'What if he as young kills himself as old?' or perhaps worse still 'What if he as old kills himself as young?'. Obviously this raises rather interesting questions concerning free will. I will however leave it to the reader to further consider this.

11.5 Warp drive

In science fiction terms like *warp drive* have become standard as a means of transportation that is in some sense faster than light. One might ask to what extent such a transportation is possible within the framework of general relativity. Well, if we have some way of preparing spacetime 'from outside the spacetime' this is certainly not a problem as depicted in Fig. 11.6.

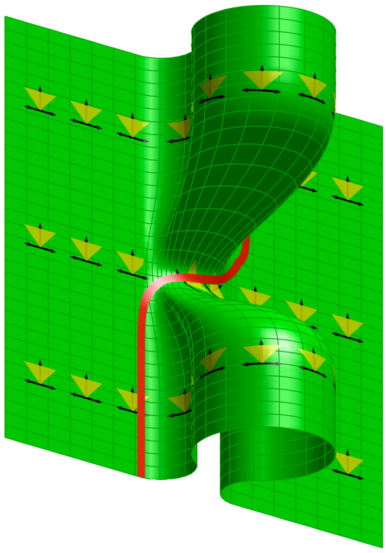


Figure 11.6: A warp drive spacetime.

As always – for any spacetime that we can come up with – there is an energy-momentum tensor that via the field equations gives the right local curvature and will automatically obey the local conservation laws of energy and momentum (via the Bianchi identities).

Assume now that in our lives everything operates as if we are living through a flat spacetime. Assume further that we have some means of freely manipulating local momentum and energy currents (and that energy may be negative etc). Then if we plan ahead we can send out emissaries that will tweak the energy momentum tensor at some preset time (according to their own clocks) and thus create from within the spacetime a warp bridge like that depicted to the right in Fig. 11.6 or like Fig 17 of

Paper II. Of course this method of transportaion is no faster than if you would join the rightmost emissary and just go there. It is different however. In the warp method you create an opportunity to travel fast should you so choose at a later time.

Of course we have assumed quite much here, the least of which is the ability to make a global smooth tweak of the energy momentum tensor – which is not practically feasible. We could perhaps hope to make a series of discrete explosions or some such. However, in the classical theory I see no problems in principle of the above type of warp drive.

Of course if we consider quantum mechanics – it is not at all clear what we can and cannot do for this case. Indeed we would need a theory of quantum gravity to exploit this fully.

We could consider a corresponding emissary scenario also for the case of an initially flat spacetime with closed space – to *at will* create a spacetime as that depicted to the right of Fig. 11.5. Again I will leave it to the reader to consider whatever paradoxes concerning free will that might occur there.

11.6 Finding generators to make a single trajectory an absolute geodesic

This section is essentially an appendix that for brevity was cut out from Paper II. The notation is therefore directly related to that of Paper II and references to equations in Paper II will be of the form 'Eq. II.xx'.

In this section we assume both the original and the absolute metric to be 2-dimensional, time independent and diagonal. For a geodesic relative to the absolute metric we have (in the coordinates for which it is diagonal):

$$\left(\frac{dx'}{dt'}\right)^2 = \frac{\bar{g}'_{tt}}{\bar{g}'_{xx}}\left(\bar{\sigma}'\bar{g}'_{tt} - 1\right) \tag{11.7}$$

Here $\bar{\sigma}'$ is a constant that is fixed for every geodesic. Suppose now that we would want *some* motion in the original metric, to be a *geodesic* in the absolute metric. It will then turn out to be practical to express the original motion using the Killing velocity, denoted by w for the motion considered. The corresponding four-velocity, prior to diagonalization, is given analogous to Eq. II.14 as:

$$q^{\mu} = \pm\sqrt{\frac{g_{tt}}{1-w^2}}\left(\frac{1}{g_{tt}}, \frac{-w}{\sqrt{-g_{xx}g_{tt}}}\right) \tag{11.8}$$

Using Eq. II.25, and Eq. II.26 we can transform q^{μ} to q'^{μ}. Then we can express $\frac{dx'}{dt'}$ as $\frac{q'^x}{q'^t}$ which is then given as a function of w. Using this is in Eq. (11.7), together with Eq. II.15 we find after simplification:

$$\frac{w^2(1-v^2)^2}{(1+v^2-2vw)^2} - \left(\bar{\sigma}' g_{tt}\frac{1+v^2}{1-v^2} - 1\right) = 0 \tag{11.9}$$

The generating velocity v has to obey this equation to make an original motion, characterized by $w(x)$, a geodesic in the absolute metric. In general it would be tricky to solve this equation for v. We can factorize it, but we still get a fourth order equation in v. There are however cases where we can deal with it analytically.

11.6.1 Considering generators

Setting $v = w$, thus demanding the generators to be absolute geodesics, Eq. (11.9) is reduced to:

$$v^2 = 1 - \bar{\sigma}' g_{tt} \tag{11.10}$$

It is easy to show that this corresponds exactly to a geodesic in the *original* spacetime. Thus we draw the conclusion that the generators of the absolute metric will be geodesics in the absolute spacetime if and only if they are geodesics in the original spacetime. While we here arrived at this conclusion assuming a Killing symmetry among other things, it is a completely general result as is shown in Appendix II.E.

11.6.2 Considering photons

Another case where Eq. (11.9) is trivialized is when we consider an outward-moving photon ($w = -1$). Then the solution is:

$$v = \frac{2 - g_{tt}\bar{\sigma}'}{2 + g_{tt}\bar{\sigma}'} \tag{11.11}$$

Here $\bar{\sigma}'$ is an arbitrary constant. Assuming a Schwarzschild line element and choosing $v = 0$ at infinity yields $\bar{\sigma}' = 2$. For this particular example we may insert the v of Eq. (11.11), and the line element of Eq. II.10, into Eq. II.15 to find:

$$\bar{g}'_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\left(1 + (1 - \frac{1}{x})^2\right) & 0 \\ 0 & \frac{2}{1+(1-\frac{1}{x})^2} \end{pmatrix} \tag{11.12}$$

We see that the absolute metric exists all the way into the singularity. We may however notice that the Killing velocity v, as defined by Eq. (11.11), becomes infinite at $x = 1/2$. That is however nothing to worry about. It just means that at this point the generators are exactly orthogonal to the Killing field. In Fig. 11.7 we see an embedding of the geometry described by Eq. (11.12).

The thick lines with arrows correspond to photons moving *outwards*. The left one, being inside the horizon, is however guided by the geometry into the singularity [4].

[4]Strictly speaking the photon trajectories should be invisible after having spiraled once around the surface since the spacetime in this embedding is layered.

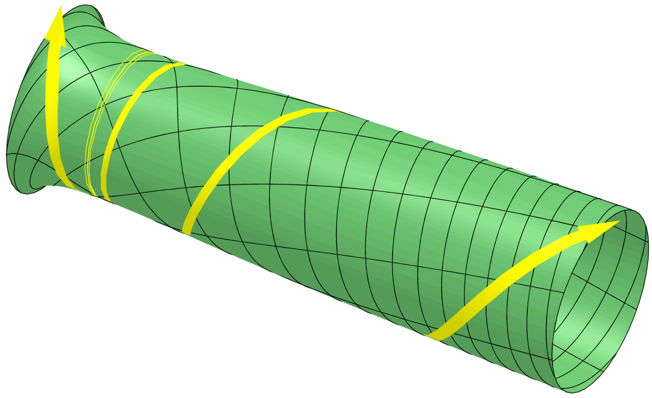


Figure 11.7: The absolute spacetime using generators $u^\mu(x)$ such that outward-moving photons follow geodesics. The radial parameter x lies in the interval $[0.55, 4]$.

The horizon is located where there is a minimum in the embedding radius. This is necessary if we want the photon to remain on a certain spatial position and at the same time follow a geodesic on the rotational surface. Also it has to be a minimum or a photon could oscillate back and forth around the horizon.

While outward-moving photons correspond to geodesics, it is apparent that inward-moving photons do not. To see this we consider an inward-moving photon at the horizon. The trajectory is there directed purely along the surface (no azimuthal variation). A geodesic tangent to the photon trajectory at this point will remain on a fix azimuthal angle, whereas the photon trajectory does not. Thus, without using any mathematics, we may understand that we cannot make both ingoing and outgoing photons correspond to geodesics, while keeping manifest time symmetry (and assuming that we want to embed the horizon). By relaxing the time independency one could hope not only to get inward- and outward-moving photons to follow geodesics, but all free particles. In Appendix II.D, II.F and II.G, we show that this is possible only in a very limited class of spacetimes.

12

Kinematical invariants

In Papers III-VIII we employ a congruence of reference worldlines that threads the spacetime[1]. The local behavior of the congruence can be described by the covariant derivative of the congruence four-velocity. This derivative can be split into different parts, known as the kinematical invariants of the congruence. They are tensors related to shear, expansion, rotation and acceleration. In this chapter we give some intuition regarding the meaning of these invariants and their relation to the covariant derivative of the congruence four-velocities.

12.1 The definitions of the kinematical invariants

The kinematical invariants of a congruence of worldlines of four-velocity u^μ are defined as (see e.g [7] p. 566):

$$
\begin{aligned}
a_\mu &= u^\alpha \nabla_\alpha u_\mu && (12.1)\\
\theta &= \nabla_\alpha u^\alpha && (12.2)\\
\sigma_{\mu\nu} &= \frac{1}{2}\left(P^\rho{}_\nu \nabla_\rho u_\mu + P^\rho{}_\mu \nabla_\rho u_\nu\right) - \frac{1}{3}\theta P_{\mu\nu} && (12.3)\\
\omega_{\mu\nu} &= \frac{1}{2}\left(P^\rho{}_\nu \nabla_\rho u_\mu - P^\rho{}_\mu \nabla_\rho u_\nu\right) && (12.4)
\end{aligned}
$$

In order of appearance these objects are denoted the acceleration vector, the expansion scalar, the shear tensor and the rotation tensor. The projection operator $P^\mu{}_\alpha$ is given by (using the $(-,+,+,+)$ convention):

$$P^\mu{}_\alpha = g^\mu{}_\alpha + u^\mu u_\alpha \tag{12.5}$$

[1]In the more intuitive Papers IV and VI, we do not explicitly talk of reference worldlines threading spacetime, but effectively they are there anyway and this chapter is relevant also for these papers.

Forming $P^{\mu}{}_{\alpha}k^{\alpha}$ yields a vector corresponding to the part of k^{μ} that is orthogonal to the congruence (as is obvious in inertial coordinates locally comoving with the congruence).

We may also introduce what we denote as the expansion-shear tensor:

$$\theta_{\mu\nu} = \frac{1}{2}\left(\nabla_{\rho}u_{\mu}P^{\rho}{}_{\nu} + \nabla_{\rho}u_{\nu}P^{\rho}{}_{\mu}\right) \tag{12.6}$$

From the normalization of u^{μ} it follows that $u^{\alpha}\nabla_{\mu}u_{\alpha} = 0$. Also we know that the covariant derivative of the metric vanishes. Then we can write Eq. (12.4) and Eq. (12.6) in a slightly different but equivalent form:

$$\omega_{\mu\nu} = \frac{1}{2}P^{\rho}{}_{\mu}P^{\sigma}{}_{\nu}(\nabla_{\sigma}u_{\rho} - \nabla_{\rho}u_{\sigma}) \tag{12.7}$$

$$\theta_{\mu\nu} = \frac{1}{2}P^{\rho}{}_{\mu}P^{\sigma}{}_{\nu}(\nabla_{\sigma}u_{\rho} + \nabla_{\rho}u_{\sigma}) \tag{12.8}$$

In this form the three-dimensional nature of the rotation tensor and the expansion-shear tensor is more obvious. From the definitions it follows that:

$$\nabla_{\nu}u_{\mu} = \omega_{\mu\nu} + \theta_{\mu\nu} - a_{\mu}u_{\nu} \tag{12.9}$$

So here are the definitions of the kinematical invariants. Now let us see if we can understand the physical meaning of these objects. In particular we will focus on the rotation and the expansion-shear tensor.

12.2 The average rotation

The rotation tensor is evidently (by its name) connected to rotation of the congruence points. The tensor is however well defined also when the congruence is shearing and deforming, in which case there is no rigid rotation. We may suspect that for this case the rotation tensor is related to an *average* rotation of the reference congruence points. With this in mind let us derive an expression for the average rotation, and see if the resulting expression is connected to the tensor $\omega_{\mu\nu}$.

Consider then a collection of moving reference points in three-dimensional Euclidean geometry. Also assume that the velocity at the origin of the coordinates in question is zero momentarily. We start by deriving an expression for the average rotation around the z-axis, considering a circle of radius r in the $z = 0$ plane. The average rotation of the points along the circle is r^{-1} times the average velocity in the counter-clockwise direction. Letting $\mathbf{u}(x, y, z)$ be the velocity of the reference points, we may express the average rotation as a line integral over the circle:

$$\omega_z = \frac{1}{r}\frac{1}{2\pi r}\int_{\circ} \mathbf{u} * \mathbf{dx} = \frac{1}{r}\frac{1}{2\pi r}\int_{\bullet} (\nabla \times \mathbf{u}) \cdot \mathbf{dS} \tag{12.10}$$

In the last equality we have used Stokes theorem, turning the line integral over the circle into a surface integral over a circular flat disc with $\mathbf{dS} = \hat{\mathbf{z}}dS$. In the limit as the radius of the circle goes to zero, $\nabla \times \mathbf{u}$ can be considered as constant (to the order necessary) and we can move it out of the integral:

$$\omega_z \simeq \frac{1}{r}\frac{1}{2\pi r}(\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}} \int dS = \frac{1}{2}(\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}} \tag{12.11}$$

So in the limit as the radius of the circle goes to zero we have:

$$\omega_z = \frac{1}{2}(\partial_x u_y - \partial_y u_x) \tag{12.12}$$

This is then the average rotation around the z-axis for the motion of the reference points in the $z = 0$ plane (in the limit where the circle over which we average goes to zero). We realize that also averaging over the the rotation considering non-zero z, Eq. (12.12) gives the local average rotation around the z-axis. Corresponding arguments gives us the average rotation around the other axes:

$$\boldsymbol{\omega} = \frac{1}{2}(\partial_y u_z - \partial_z u_y, \partial_z u_x - \partial_x u_z, \partial_x u_y - \partial_y u_x) \tag{12.13}$$

So here we have the average rotation vector that we set out to find. For some further intuition see Fig. 12.1.

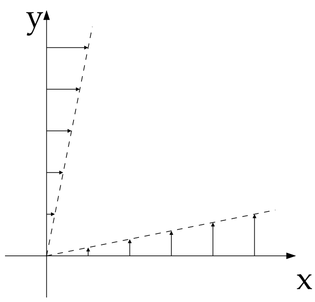


Figure 12.1: A shearing velocity field in two dimensions. The reference points along the x-axis are rotating counter-clockwise whereas the points along the y-axis have a clockwise rotation. The average of the corresponding angular velocities (counted positive in the counter-clockwise direction) is ω_z, which for this case is zero.

12.2.1 A matrix formulation of rotation

Forming $\boldsymbol{\omega} \times \boldsymbol{x}$ we get the contribution coming from an (average) rotation to the velocity at a point $\mathbf{x}$:

$$\boldsymbol{\omega} \times \boldsymbol{x} = (\omega_y z - \omega_z y, \;\; \omega_z x - \omega_x z, \;\; \omega_x y - \omega_y x) \tag{12.14}$$

Now form a matrix ω_{ij}:

$$\omega_{ij} = \frac{1}{2}(\partial_j u_i - \partial_i u_j) \tag{12.15}$$

In matrix form this becomes:

$$\omega_{ij} : \frac{1}{2}\begin{bmatrix} 0 & \partial_y u_x - \partial_x u_y & \partial_z u_x - \partial_x u_z \\ \partial_x u_y - \partial_y u_x & 0 & \partial_z u_y - \partial_y u_z \\ \partial_x u_z - \partial_z u_x & \partial_y u_z - \partial_z u_y & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{12.16}$$

It follows that we have:

$$\omega^i{}_j x^j = (\omega_y z - \omega_z y, \omega_z y - \omega_y z, \omega_x y - \omega_y x) \tag{12.17}$$

The right hand side of this equation is identical to the right hand side of Eq. (12.14). So we understand that $\omega^i{}_j x^j$, where $\omega^i{}_j$ is defined by Eq. (12.15), gives the velocity of the congruence point at x^i coming from an average rotation.

It is not hard to realize that we can form the rotation tensor as defined by Eq. (12.15) from the rotation vector as [2]:

$$\omega^{ij} = -\epsilon^{ijk}\omega_k \tag{12.18}$$

Here ϵ^{ijk} is defined by:

$$\epsilon^{ijk} = \begin{cases} +1 & : \quad ijk \text{ even permutation of } 123 \\ -1 & : \quad ijk \text{ odd permutation of } 123 \\ 0 & : \quad ijk \text{ some indices equal} \end{cases} \tag{12.19}$$

The inverse of Eq. (12.18) is simply [3]:

$$\omega^i = -\frac{1}{2}\epsilon^{ijk}\omega_{jk} \tag{12.20}$$

So now we have some understanding for how the average rotation is related to the derivatives of the velocity field.

[2]Multiplying the right hand side of Eq. (12.18) by a factor $(-\mathrm{Det}(g_{ij}))^{-\frac{1}{2}}$ (see [4] p. 98-99 concerning tensor densities) this is a covariant relation where ω_{ij} is minus the *dual* of ω_i (see [7] p. 88).

[3]We may guess this and then check it. Alternatively we could multiply Eq. (12.18) by ϵ_{mij}, using the fact that $\epsilon_{mij}\epsilon^{ijk} = 2\delta_m{}^k$

12.3 About deformation

We can split the velocity derivative tensor into a symmetric and an antisymmetric part. We have then to first order in x^i:

$$\begin{aligned} u_i &= x^j \partial_j u_i && (12.21)\\ &= x^j \left(\frac{1}{2}(\partial_j u_i - \partial_i u_j) + \frac{1}{2}(\partial_j u_i + \partial_i u_j)\right) && (12.22)\\ &= x^j \left(\omega_{ij} + \theta_{ij}\right) && (12.23) \end{aligned}$$

Here we have introduced θ_{ij} as:

$$\theta_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{12.24}$$

Now we would like to show that this object is related to deformations (of any shape connected to the congruence points). It seems obvious that the most general local motion that is non-deforming corresponds to a rigid rotation (as regards the momentary velocities). For this kind of motion we must have (there exists an $\boldsymbol{\omega}$ such that):

$$\mathbf{u} = \boldsymbol{\omega} \times \boldsymbol{x} \tag{12.25}$$

Obviously in this case the motion due to the average rotation is precisely the motion – and the motion due to the average rotation is $u^k = \omega^k{}_j x^j$. From Eq. (12.23) it then follows that we must have $x^j \theta_{ij} = 0$ for all x^j. This can only happen if $\theta_{ij} = 0$. So it follows that rigid motion implies vanishing θ_{ij} [4]. Conversely, given that $\theta_{ij} = 0$, it follows from Eq. (12.23) that $u^i = \omega^i{}_j x^i$. Since $\omega^i{}_j$ is an antisymmetric tensor it follows from Eq. (12.14), Eq. (12.16) and Eq. (12.17) that this corresponds to a rigid rotation (with rotation vector given by Eq. (12.16)). So we conclude that the congruence is rigid if and only if $\theta_{ij} = 0$.

A simpler argument would be to say that $\theta^i{}_j x^j$ gives the extra velocity apart from that coming from an average (best fit) rotation. Hence the congruence is rigid if and only if $\theta_{ij} = 0$.

12.4 Back to four-dimensional formalism

Connecting the three-dimensional ω_{ij} and θ_{ij} to their four-dimensional analogies $\omega_{\mu\nu}$ and $\theta_{\mu\nu}$ is very simple. In freely falling coordinates locally comoving with the congruence, the spatial parts of $\omega_{\mu\nu}$ and $\theta_{\mu\nu}$ precisely equals their three-dimensional analogies. Moreover, in these coordinates all time components of $\omega_{\mu\nu}$ and $\theta_{\mu\nu}$ vanishes due

[4]Alternatively we could just evaluate $\partial_i u_j + \partial_j u_i$ for $u_i = \epsilon_i{}^{jk}\omega_j x_k$. Here ω_j is to be treated as a constant (to the necessary order). Using $\partial_i x_j = \delta_{ij}$ and the antisymmetry of ϵ_{ijk} we readily find that $\theta_{ij} = 0$.

to the projection. It follows that the congruence is rigid if and only if $\theta_{\mu\nu}$ vanishes, independent of what coordinates we are using. Also in other respects the meanings are analogous. In particular forming $\omega^{\mu}{}_{\nu}\Delta x^{\mu}$ (for some vector Δx^{μ} orthogonal to u^{μ}) gives that part of the velocity of the congruence point at the position Δx^{μ} (relative to the inertial system whose origin momentarily comoves with the congruence) that can be seen as coming from an average rotation.

Concerning the expansion scalar θ it is not hard to realize that this corresponds to the relative volume increase (of a box of reference points) per unit time. For instance $\theta = 0.1s^{-1}$ implies a momentary relative volume increase rate of 10% per second. Notice that we can have $\theta = 0$ while for instance having expansion in the x-direction and contraction in the y-direction.

Having understood the meaning of $\theta_{\mu\nu}$ and θ we understand that the shear tensor $\sigma_{\mu\nu}$, as defined by Eq. (12.3), describes that part of the local congruence motion that is neither due to an (average) rotation nor an (average) expansion.

As regards the remaining kinematical invariant a^{μ}, it is simply the acceleration of the congruence.

12.4.1 The four dimensional analogue to the rotation vector

In the three-dimensional analysis we saw in Eq. (12.20) how we could relate ω^i to ω_{ij}:

$$\omega^i = -\frac{1}{2}\epsilon^{ijk}\omega_{jk} \tag{12.26}$$

We would now like to have a corresponding four-covariant expression. In particular we would like a tensorial expression for a four vector corresponding to $(0, \boldsymbol{\omega})$ in freely falling coordinates locally comoving with the congruence. A natural extension of Eq. (12.26) is:

$$\omega^{\mu} = \frac{1}{2}\frac{1}{\sqrt{g}}u_{\sigma}\epsilon^{\sigma\mu\gamma\rho}\omega_{\gamma\rho} \tag{12.27}$$

Here $g = -\mathrm{Det}(g_{\alpha\beta})$ (see [4] p. 98-99 concerning tensor densities). Conversely we have in analogy with Eq. (12.18):

$$\omega^{\mu\nu} = \frac{1}{\sqrt{g}}u_{\sigma}\epsilon^{\sigma\mu\nu\rho}\omega_{\rho} \tag{12.28}$$

So now we have covariant four-tensor equations relating the rotation tensor to the vector describing the local average rotation.

13

Lie transport and Lie-differentiation

To form a covariant derivative of a vector defined along a worldline we need a way of transporting the vector, according to some law of transport, from one point to another along the worldline so that we can form the difference between the actual vector and the transported vector (at a single point). This general idea of differentiation is relevant for papers III-VIII – concerning inertial forces and gyroscope precession.

There are a few (standard) means of transporting a vector in general relativity. There is the ordinary parallel transport where, relative to freely falling coordinates, the components of the vector does not change as we move it along the worldlines. Then there is the Fermi-Walker transport, describing how the spin four-vector is transported along a worldline. Lastly there is the Lie transport. This transport is defined with respect to a reference congruence, assuming the worldline along which we transport the vector is directed along the congruence. In this chapter we give a little background to the concept of contravariant and covariant Lie differentiation.

13.1 Contravariant Lie differentiation

Suppose that we have an arbitrary contravariant vector field u^μ. We then choose coordinates such that $u^\mu = \delta^\mu{}_0$ (assuming the vector field to be reasonably well behaved in the region in question). Note that this procedure has nothing to do with geometry or Lorentz structure. The idea is illustrated in Fig. 13.1.

The induced time-slices (although strictly speaking what we are doing need not have anything to do with time) are uniquely defined by the procedure, given an arbitrary initial slice (and labeling of this slice). The labeling of the congruence lines (the streamlines of the vector field) is completely arbitrary. In these coordinates we now want to transport an arbitrary vector K^μ along a congruence line such that the coordinate derivative of the vector vanishes, see Fig. 13.2.

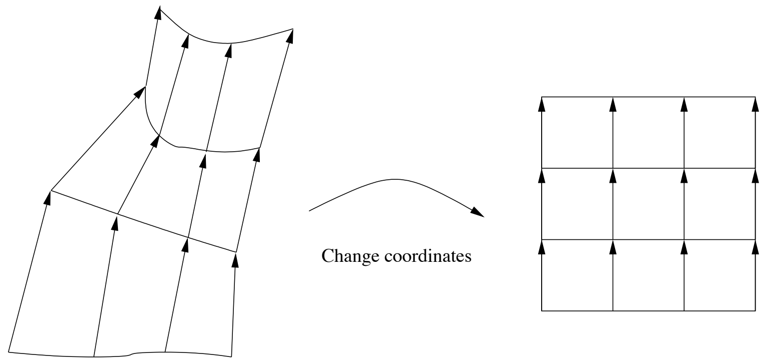


Figure 13.1: Adapting coordinates to a (contravariant) vector field

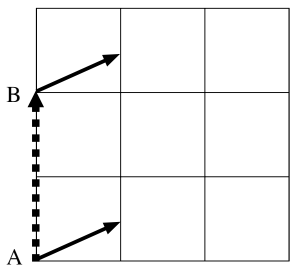


Figure 13.2: Transporting a vector in the preferred coordinates

We understand that the transport of this vector is uniquely determined by the procedure above (how we label the congruence lines does not matter)[1]. Consider now K^μ to be a field that is transported into itself as outlined above. We have in the coordinates in question (for the moment we will assume the existence of an affine connection):

$$\begin{aligned} \partial_\alpha u^\mu &= 0 && (13.1)\\ \nabla_\alpha u^\mu &= \Gamma^\mu_{\alpha\beta} u^\beta && (13.2) \end{aligned}$$

Here ∇_μ is the covariant derivative. Also we have in the coordinates in question:

$$\begin{aligned} 0 &= u^\alpha \partial_\alpha K^\mu && (13.3)\\ &= u^\alpha \nabla_\alpha K^\mu - u^\alpha \Gamma^\mu_{\alpha\beta} K^\beta && (13.4)\\ &= u^\alpha \nabla_\alpha K^\mu - (\nabla_\beta u^\mu) K^\beta && (13.5) \end{aligned}$$

So we have our transport equation in covariant form:

$$u^\alpha \nabla_\alpha K^\mu = K^\alpha \nabla_\alpha u^\mu \qquad (13.6)$$

[1] Also we may understand that what initial arbitrary time slice we are considering does not matter either. If we choose a different slicing and draw these slices in the original coordinates as depicted in Fig. 13.2, they will be tilted – but the tilt must be the same everywhere along a single congruence line. We may thus understand that the time component of the vector in question will be affected, but the transport will not be affected.

Actually we do not need to assume an affine connection. In the particular coordinates in question we have:

$$\left.\begin{array}{lll} u^\alpha \partial_\alpha K^\mu & = & 0 \\ \\ \partial_\alpha u^\mu & = & 0 \end{array}\right\} \quad \Rightarrow \quad u^\alpha \partial_\alpha K^\mu = K^\alpha \partial_\alpha u^\mu \tag{13.7}$$

The latter equation is through its form such that it holds in all coordinates given that it holds in one system (do the transformation to another system and we find that the extra, non-tensorial, terms cancel). Assuming then that we have a metrical structure and a covariant derivative, equation Eq. (13.6) follows immediately:

Also for a bit of extra intuition, assuming that we have a metrical structure, have a look at Fig. 13.3.

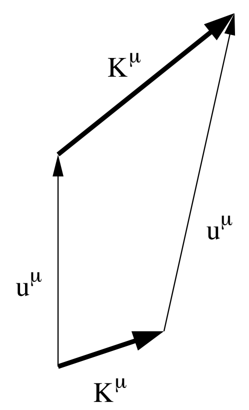


Figure 13.3: Transporting a vector K^μ as seen from freely falling coordinates.

It is obvious that the change of K^μ should depend on the change of u^μ in the direction of K^μ. Our first guess from this point of view would likely be:

$$\frac{DK^\mu}{D\tau} = K^\alpha \nabla_\alpha u^\mu \tag{13.8}$$

This is in fact exactly Eq. (13.6), so the transport equation is very intuitively reasonable. Also, given an arbitrary vector field K^μ, we can define the Lie derivative of this field as:

$$L_u K^\mu = u^\alpha \nabla_\alpha K^\mu - K^\alpha \nabla_\alpha u^\mu \tag{13.9}$$

We can also express this in terms of the kinematical invariants of the congruence. We have:

$$\nabla_\alpha u^\mu = \omega^\mu{}_\alpha + \theta^\mu{}_\alpha - a^\mu u_\alpha \tag{13.10}$$

In particular, for spatial K^μ, we have then:

$$L_u K^\mu = u^\alpha \nabla_\alpha K^\mu - (\omega^\mu{}_\alpha + \theta^\mu{}_\alpha) K^\alpha \tag{13.11}$$

13.2 Covariant Lie differentiation

We may perform a similar analysis considering a covariant vector field K_μ. We analogously demand that the components of this vector should be unaltered as we move it along the congruence in the preferred coordinates. Analogously to Eq. (13.7) we have

$$\left.\begin{array}{lcl} u^\alpha \partial_\alpha K_\mu & = & 0 \\ \\ \partial_\alpha u^\mu & = & 0 \end{array}\right\} \qquad \rightarrow \qquad u^\alpha \partial_\alpha K_\mu = -K_\alpha \partial_\mu u^\alpha \tag{13.12}$$

We note that we have inserted a minus sign here as compared to Eq. (13.7). Whereas the equality holds equally well independently of what sign we choose in the preferred coordinates, the sign is chosen such that the equality holds in *any* coordinate system. To prove this we evaluate $u'^\alpha \frac{\partial}{\partial x'^\alpha} K'_\mu + K'_\alpha \frac{\partial}{\partial x'^\mu} u'^\alpha$, using the definitions of how the covariant and contravariant objects are related to their non-primed versions. We also use the trick of differentiating a Kronecker delta in the form:

$$0 = \frac{\partial}{\partial x'^\mu} \left(\frac{\partial x'^\alpha}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\alpha} \right) \tag{13.13}$$

Expanding this derivative and using the resulting expression, we readily find that $u^\alpha \frac{\partial}{\partial x^\alpha} K^\mu + K^\alpha \frac{\partial}{\partial x^\alpha} u^\mu$ is in fact a tensor (although its individual terms are not). Hence if it vanishes in one system it vanishes in all systems. It follows that a covariant vector field Lie-dragged into itself obeys:

$$u^\alpha \partial_\alpha K_\mu = -K_\alpha \partial_\mu u^\alpha \tag{13.14}$$

Note that it is not only the sign that differs from the analogous equation Eq. (13.7) of the preceding section. How the summation runs is also different. For this case the form is less intuitive.

Lie differentiation of a covariant field is then given by:

$$L_u K_\mu = u^\alpha \nabla_\alpha K_\mu + K_\alpha \nabla_\mu u^\alpha \tag{13.15}$$

Note that here we are using covariant differentiation (to make the derivative manifestly covariant), but the form is such that we could instead use ordinary differentiation.

Notice in particular that if we have the Lie derivative of a covariant field we do not get the Lie derivative of the corresponding contravariant field by raising the former with the metric:

$$g^{\mu\alpha} L_u K_\alpha \neq L_u K^\mu \tag{13.16}$$

In other words Lie-differentiation does not in general commute with the raising and lowering of indices [2].

[2] If the vector field obeys $\nabla_\alpha u^\mu = 0$, Lie differentiation does however commute with lowering and raising of indices.

13.3 Comments

One might consider other transports of the vector in question, not just along the congruence, as depicted in Fig. 13.4

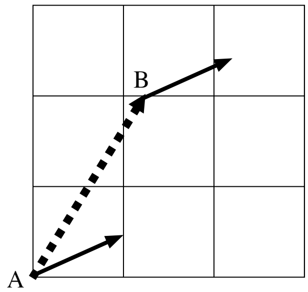


Figure 13.4: Trying a different transport equation with respect to a reference congruence.

In a particular set of coordinates we may here demand that the components of the vector should be unaltered as we transport it. For this case it is however obvious that the labeling of the congruence lines will affect the transport.

To gain intuition concerning the difference between the raised covariant transport and the ordinary contravariant transport, let us study a particular example in 2+1-dimensional special relativity. Let $K_\mu = \partial_\mu \phi$ where $\phi = x$ in inertial coordinates (t, x, y). Now consider another coordinate system that is shearing relative to the inertial coordinates such that $x' = x$ as depicted in Fig. 13.5.

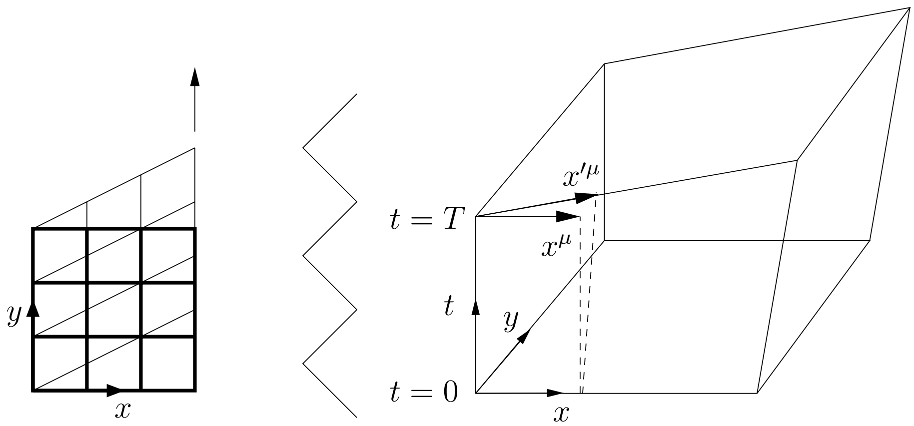


Figure 13.5: To the left the shearing coordinates (thin lines) relative to the inertial coordinates (thick lines) at a time $t = T$. To the right the shearing coordinates in a 2+1 spacetime illustration seen relative to the inertial coordinates.

We have in the inertial coordinates $\partial_\mu \phi = (0, 1, 0)$. Also in the shearing coordinates we have $\partial'_\mu \phi = (0, 1, 0)$. Now consider Lie transporting (with respect to the shearing coordinates) a contravariant vector directed in the x-direction at $t = 0$ along the congruence. At $t = T$ it will be directed in the x'^μ direction (see Fig. 13.5). On the other hand, if we consider a covariant transport of the same vector then in the shearing coordinates (0,1,0) will go to (0,1,0) (per definition), but the latter vector – when raised with the metric – we know has the meaning of the gradient of ϕ which is directed in the x^μ direction. Thus relative to the shearing coordinates the raised covariantly transported vector will rotate. So here we have an intuitive example illustrating the difference between the contravariant Lie transport and the raised covariant Lie transport.

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